AUG 22 2022

Lattice Monotonicity Theorems for Height f^n's

Goal: Understand height-functions w/ new tools.

An example: The $\mathbb{Z}$GF.

$G = (V, E)$ finite graph

$\varphi : V \to \mathbb{Z}$ random field, assoc. partition f^n:

$$Z_\lambda^{\mathbb{Z}GF, G} := \sum_{\varphi : V \to \mathbb{Z} : \varphi|_{\partial V} = 0} \exp\left(-\frac{\lambda}{2} \langle \varphi, -\Delta \varphi \rangle\right)$$

w/ $\langle \varphi, -\Delta\varphi \rangle \equiv \sum_{\{x,y\} \in E} (\varphi_x - \varphi_y)^2$.

Def.: A "lattice" in $\mathbb{R}^n$ is a set

$$\mathcal{L} := \{ L v \mid v \in \mathbb{Z}^n \}$$

where $L \in \mathrm{Mat}_{n \times n}(\mathbb{R})$ specifies the lattice.

Example: ⊛ $L = \mathbb{1}_{n \times n} \rightsquigarrow \mathcal{L} = \mathbb{Z}^n$

$$\circledast \quad \left.\begin{matrix} n=1 \\ L=2 \end{matrix}\right\} \mathcal{L} = 2\mathbb{Z}.$$

For any lattice $\mathcal{L}$ and $A \in \mathrm{Mat}_{n\times n}(\mathbb{R}) : A \geq 0$,

define the random variable $\psi \in \mathcal{L}$ via

the assoc. part. f^n:

$$Z_{A,\mathcal{L}} := \sum_{\psi \in \mathcal{L}} e^{-\frac{1}{2}\langle \psi, A\psi\rangle_{\mathbb{R}^n}}$$

Cf.: $Z_{A,\mathbb{R}^n} \equiv \int_{\psi \in \mathbb{R}^n} d\psi \, e^{-\frac{1}{2}\langle \psi, A\psi\rangle_{\mathbb{R}^n}}$

Using ψ one may obtain:

① The $\mathbb{Z}$GF on $G = (V, E)$

Let $n := |V|$.

Set up a bij. $\eta : V \xrightarrow{\sim} \{1, \dots, n\}$.

Induces a lin. iso. $\hat{\eta} : \ell^2(V) \xrightarrow{\sim} \mathbb{R}^n$.

$$A := -\lambda \hat{\eta} \Delta \hat{\eta}^{-1}$$

$$L := \hat{\eta} \underbrace{\chi_{\partial v c}(X)}_{\text{Dirichlet B.C.}} \hat{\eta}^{-1}$$

Can also instead choose:

$P :=$ proj. onto $\ker(-\Delta)$

$\equiv$ ——— $\{\varphi : \sum_{x\in V}{}' \varphi_x = 0\}$

$$L = \hat{\eta}\, P^{\perp}\, \hat{\eta} .$$

② The Coulomb gas on $G = (V,E)$

Same as before, but $A = \frac{(2\pi)^2}{\lambda}(-\Delta)^{-1}$.

Basic object of study: M.G.F.

$\forall\ v \in \mathbb{R}^n$,

$$M_{A,\mathcal{X}}[v] := \mathbb{E}_{A,\mathcal{X}}\left[\exp(\langle v, \psi\rangle_{\mathbb{R}^n})\right].$$

Interesting because monotonicity of MGF implies monotonicity of 2^{nd} moment, e.g..

Lemma (RSD):

$$M[u]M[v] \leq \sqrt{M[u+v]\, M[u-v]}$$

Proof: $\mathcal{X}^2 := \mathcal{X} \oplus \mathcal{X}$

$$M[u]M[v] = Z_{A,\mathcal{X}}^{-2} \sum_{\psi\in\mathcal{X}^2}{}' e^{-\frac{1}{2}\langle\psi, A\oplus A\psi\rangle + \langle \left[\begin{smallmatrix}u\\v\end{smallmatrix}\right], \psi\rangle}$$

$$U := \frac{1}{\sqrt{2}} \underbrace{\begin{bmatrix} \mathbb{1}_n & \mathbb{1}_n \\ \mathbb{1}_n & -\mathbb{1}_n \end{bmatrix}}_{=:T} \quad \text{is unitary}$$

$$\Rightarrow \langle \psi, A\oplus A \psi \rangle = \langle U\psi, U A\oplus A U^* U \psi \rangle$$

$$\overset{[U, A\oplus A]=0}{=} \langle U\psi, A\oplus A\, U\psi \rangle$$

$$= \frac{1}{2} \langle T\psi, A\oplus A T\psi \rangle$$

$$\left\langle \begin{bmatrix} u \\ v \end{bmatrix}, \psi \right\rangle = \left\langle U \begin{bmatrix} u \\ v \end{bmatrix}, U\psi \right\rangle$$

$$= \frac{1}{2} \left\langle \begin{bmatrix} u+v \\ u-v \end{bmatrix}, T\psi \right\rangle$$

$$\sum_{\psi \in \mathbb{Z}^2} e^{-\frac{1}{4}\langle T\psi, A\oplus A T\psi \rangle - \frac{1}{2}\left\langle \begin{bmatrix} u+v \\ u-v \end{bmatrix}, T\psi \right\rangle} =$$

$$= \sum_{\psi \in T\mathbb{Z}^2} e^{-\frac{1}{4}\langle \psi, A\oplus A\psi \rangle - \frac{1}{2}\left\langle \begin{bmatrix} u+v \\ u-v \end{bmatrix}, \psi \right\rangle} =: (*)$$

But $T\mathbb{Z}^2 = \bigsqcup_{w \in \mathbb{Z}/2\mathbb{Z}} (2\mathbb{Z}+w)^2.$

$$\Rightarrow (*) = \sum_{w \in \mathbb{Z}/2\mathbb{Z}} \sum_{\psi \in (2\mathbb{Z}+w)^2} e^{-\frac{1}{4}\langle \psi, A\oplus A\psi \rangle - \frac{1}{2}\left\langle \begin{bmatrix} u+v \\ u-v \end{bmatrix}, \psi \right\rangle}$$

$$= \sum_{w \in \mathbb{Z}/2\mathbb{Z}} \prod_{z \in \{u \pm v\}} \sum_{\psi \in 2\mathbb{Z}+w} e^{-\frac{1}{4}\langle \psi, A\psi \rangle - \frac{1}{2}\langle z, \psi \rangle}$$

$\Rightarrow$ When $v=0$ we get

$$\mathbb{M}[u] = Z_{A,\mathcal{L}}^{-2} \sum_{w\in\mathcal{L}/2\mathcal{L}} \left(\sum_{\psi\in 2\mathcal{L}+w} e^{-\frac{1}{4}\langle\psi,A\psi\rangle + \frac{1}{2}\langle u,\psi\rangle} \right)^2$$

Applying Cauchy-Schwarz on $\circledast$ we get:

$$\circledast \leq \prod_{z\in\{u\pm v\}} \left(\sum_{w\in\mathcal{L}/2\mathcal{L}} \left(\sum_{\psi\in 2\mathcal{L}+w} e^{-\frac{1}{4}\langle\psi,A\psi\rangle + \frac{1}{2}\langle z,\psi\rangle} \right)^2 \right)^{1/2}$$

$\blacksquare$

Def.: Let $\mathcal{M}, \mathcal{L}$ be two lattices. We say $\mathcal{M}$ is a sub-lattice of $\mathcal{L}$ iff $\mathcal{M}\subseteq\mathcal{L}$ as s/sets of $\mathbb{R}^n$.

Example: $(2\mathbb{Z})^n \subseteq \mathbb{Z}^n$.

Thm. (RSD): If $\mathcal{M}\subseteq\mathcal{L}$ then

$$\mathbb{M}_{A,\mathcal{M}}[v] \leq \mathbb{M}_{A,\mathcal{L}}[v] \qquad (v\in\mathbb{R}^n)$$

Proof: Write $\mathcal{L} = \coprod_{w\in\mathcal{L}/\mathcal{M}} \mathcal{M}+w$.

$$Z_{\mathcal{M}} Z_{\mathcal{L}} \mathbb{M}_{\mathcal{M}}[v] =$$

$$= \Big(\sum_{w \in \mathcal{L}/\mathcal{M}} \sum_{\psi \in \mathcal{M}+w} e^{-\frac{1}{2}A(\psi)} \Big) Z_{\mathcal{M}} \mathbb{M}_{\mathcal{M}}[v]$$

$$= Z_{\mathcal{M}}^2 \sum_{w \in \mathcal{L}/\mathcal{M}} e^{-\frac{1}{2}A(w)} \underbrace{\mathbb{M}_{\mathcal{M}}[-Aw]\, \mathbb{M}_{\mathcal{M}}[v]}$$

$$\leq \sqrt{\mathbb{M}_{\mathcal{M}}[-Aw+v]\, \mathbb{M}_{\mathcal{M}}[-Aw-v]}$$

$$\leq \frac{1}{2}\Big(\mathbb{M}_{\mathcal{M}}[-Aw+v] + \mathbb{M}_{\mathcal{M}}[-Aw-v] \Big)$$

$\mathcal{M} \mapsto -\mathcal{M}$ symm.

$$= \mathbb{M}_{\mathcal{M}}[-Aw+v]$$

$$= Z_{\mathcal{M}} Z_{\mathcal{L}} \mathbb{M}_{\mathcal{L}}[v].$$

∎

<u>Corollary</u>: (Gaussian domination)

$$\mathbb{M}_{\lambda}^{\mathbb{Z}GF, G}[v] \leq \mathbb{M}_{\lambda}^{GFF, G}[v] \qquad (v \in \mathbb{R}^V)$$

<u>Proof</u>:

$$\mathbb{M}_{\lambda}^{\mathbb{Z}GF, G}[v] = \mathbb{M}_{-\lambda\Delta, \mathbb{1}^{\perp} \mathbb{Z}^{|V|}}[v]$$

$$\mathbb{M}_{\lambda}^{GFF, G}[v] = \lim_{j \to \infty} \mathbb{M}_{-\lambda\Delta, \mathbb{1}^{\perp}(2^{-j}\mathbb{Z})^{|V|}}[v]$$

Since $\mathbb{P}^{\perp}(2^{-j}\mathbb{Z})^{|V|} \subseteq \mathbb{P}^{\perp}(2^{-\tilde{j}}\mathbb{Z})^{|V|}$ whenever $\tilde{j} \geq j$, we get the result.

<u>Lemma (RSD):</u> w/ $\mathbb{H}F$ the Hessian of a function $F: \mathbb{R}^n \to \mathbb{R}$,

$$(\mathbb{H}F)_{ij}(x) \equiv (\partial_i \partial_j F)(x)$$

$$(\nabla F)_i(x) \equiv (\partial_i F)(x)$$

$$\frac{1}{M[u]} \mathbb{H}M[u] \geq \mathbb{H}M[0] + M[u]^{-2} \nabla M[u] \otimes (\nabla M[u])^*$$

<u>Proof</u>: $F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$F(u,v) := M[u+v]M[u-v] - M[u]^2 M[v]^2 \geq 0$$

$$F(u,0) = 0 \longrightarrow v=0 \text{ is a min. } \forall u$$

$$\longrightarrow (\mathbb{H}F)(u,0) \geq 0$$

But $\frac{1}{2}\mathbb{H}F(u,0) = M[u]\mathbb{H}M[u] - M[u]^2 \mathbb{H}M[0]$

$$- \nabla M[u] \otimes \nabla M[u]^* .$$

Corollary: $\mathbb{E}[e^{\langle u,\psi\rangle}\langle\psi,M\psi\rangle] \geq \mathbb{M}[u]\mathbb{E}[\langle\psi,M\psi\rangle]$

for any $M \geq 0$.

Proof: $H\mathbb{M}[u] = \mathbb{E}[e^{\langle u,\psi\rangle}\psi\otimes\psi^*]$

$\nabla\mathbb{M}[u] = \mathbb{E}[e^{\langle u,\psi\rangle}\psi]$

$tr(MN) \geq 0 \quad \forall\ M,N \geq 0$

Corollary: $\forall\ A \leq B$

$$\mathbb{M}_{B,\mathcal{X}}[v] \leq \mathbb{M}_{A,\mathcal{X}}[v] \qquad (v\in\mathbb{R}^n)$$

Proof: Define the homotopy

$$[0,1] \ni t \longmapsto tA + (1-t)B =: A_t.$$

W.T.S. $\partial_t \mathbb{M}_{A_t,\mathcal{X}}[v] \leq 0.$

But

$$2\partial_t\mathbb{M}_{A_t,\mathcal{X}}[v] = \mathbb{E}_{A_t,\mathcal{X}}[\langle\psi,(A-B)\psi\rangle]\mathbb{M}_{A_t,\mathcal{X}}[v]$$
$$-\mathbb{E}_{A_t,\mathcal{X}}[\langle\psi,(A-B)\psi\rangle e^{\langle u,\psi\rangle}] \leq 0.$$

Corollary: $(0,\infty) \ni \lambda \longmapsto \mathbb{E}_{\lambda}^{\nabla LGF,G}[(n_x-n_y)^2]$

is monotone decreasing.

Minorization

Lammers' new deloc. proof only works for graphs of degree 3.

How to generalize it?

Lattice monotonicity & "Minorization".

<u>Thm.</u>: Let $G = (V,E)$ be a doubly periodic planar graph of degree $d > 3$.
Then $\exists$ graph $\tilde{G}_d$ of max. deg. 3 s.t.

$$\mathbb{M}_{\lambda}^{\mathbb{Z}GF, G}[v] \geq \mathbb{M}_{f(\lambda,d)}^{\mathbb{Z}GF, \tilde{G}}[v].$$

<u>Proof</u>: ① Divisibility property for $\mathbb{Z}GF$:

$$\exp\left(-\tfrac{1}{2}\lambda (n_x - n_y)^2\right) = \sqrt{\frac{2\lambda}{\pi}} \int_{\psi_{xy} \in \mathbb{R}} d\psi_{xy}\, e^{-\lambda\left[(n_x - \psi_{xy})^2 + (n_y - \psi_{xy})^2\right]}$$

② May convert each $\psi_{xy} \in \mathbb{Z}$ to $\psi_{xy} \in \mathbb{R}$

via Gaussian dom. above.

This would only lower the MGF.

③ May take coupling on vertices

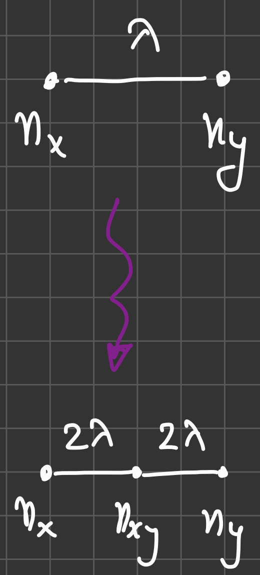
$\lambda \mapsto 2\lambda$

to get homogeneous couplings.

This would only lower the MGF.

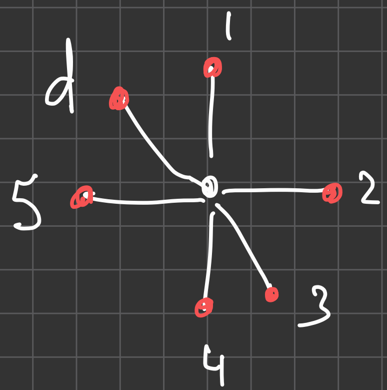
Note that only adding vertices alone w/o modifying couplings would heighten the MGF!

<u>Conclusion</u>:



lowers MGF.

If we add a mid-point on <u>every</u> edge of G, then each original edge is now the center of a star-graph, and this tiles our new graph:



S_d

Important: Identifying vertices (= sub-lattice monotonicity) makes MGF go down.

Idea: By consecutively splitting edges and identifying vertices, S_d may be replaces by a tree of max deg. 3 whose root is @ the original vertex and whose leaves are the leaves of S_d.

Alogorithm: $\ell := \lceil \log_2(d) \rceil$

① div. each edge of S_d into ℓ sub-edges to get a tree $S_{d,\ell}$.

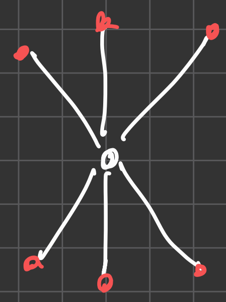
② All closest vertices to the root of $S_{d,\ell}$ are grouped into two and each group identified.

② Repeat w/ NNN of root

③ Continue $\ell-1$ times, possibly avoiding some restrictions if $d<2^{\ell}$.

$\blacksquare$

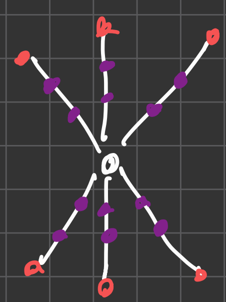
Example



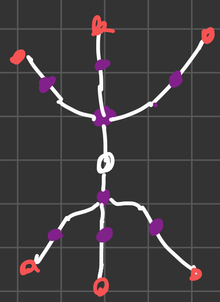
S_d

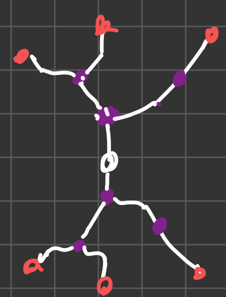
$d=6$

$\ell=3$



$S_{d,\ell}$





Final tree

Open Question: Spectral lattice monotonicity

Let $L, M \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ s.t.

$$L \geqslant M$$

(Note: L, M S.A. NOT necessary for concept of lattice)

This does NOT imply that $\mathcal{L} \subseteq \mathcal{M}$

unless $L = \alpha M \quad \exists\ \alpha \in \mathbb{N}_{\geq 1}$.

Nonetheless, is it true that

$$\mathbb{M}_{A,\mathcal{L}}[v] \leqslant \mathbb{M}_{A,\mathcal{M}}[v] \quad ?$$

Why?: The spectrally-smaller the lattice, the less severe the restriction, the larger the fluctuations.

Why is it interesting?

$$\lambda\, \mathbb{E}^{\mathbb{Z}\mathrm{GF}}_{\lambda}[\langle v,\varphi\rangle^2] + \frac{(2\pi)^2}{\lambda}\mathbb{E}^{\mathrm{CG}}_{\frac{(2\pi)^2}{\lambda}}[\langle v,-\Delta^{-1}\varphi\rangle^2] = \mathbb{E}^{\mathrm{GFF}}_{1}[\langle v,\varphi\rangle^2]$$

may be recast as

$$A_\lambda := \sqrt{\frac{-\lambda\Delta}{2\pi}}, \quad \mathcal{L} := \sqrt{2\pi}\, P_0^\perp \mathbb{Z}^V$$

$\uparrow$ away from ker $-\Delta$.

$$\mathbb{E}_{\mathbb{1}, A_\lambda \mathcal{L}}[\langle v,\psi\rangle^2] + \mathbb{E}_{\mathbb{1}, A_\lambda^{-1}\mathcal{L}}[\langle v,\psi\rangle^2] = \|v\|^2.$$

If $\lambda \ll 1$, $A_\lambda \le A_\lambda^{-1}$ thanks to RG

JKKN'77 get $\lambda_c = \frac{\pi}{2}$.

$\sigma(-\Delta) = [0, 4d]$.

$\rightsquigarrow$ for $d=2$, $\quad 8 \cdot \frac{\lambda_c}{2\pi} \overset{!}{=} \frac{2\pi}{8\lambda_c} \rightsquigarrow \boxed{\lambda_c = \frac{\pi}{4}}$.

≈ 0.785.

Numerical $\lambda_c = 0.74..$

Annealed RSD Theory – ZUF

DEF.: (Annealed Gauss. int.)

$U : \mathbb{Z} \to \mathbb{R}$ is an annealed Gaussian int.

iff $\exists$ non-neg. Borel msr. μ_U on $[0,\infty)$:

$$e^{-U(q)} = \int_{\lambda=0}^\infty e^{-\frac{1}{2}\lambda q^2}\, d\mu_U(\lambda) \qquad (q \in \mathbb{Z})$$

Random height f^n $\varphi : V \to \mathbb{Z}$ w/ assoc.

part. f^n

$$Z^{\mathbb{Z}UF, G} := \sum_{\varphi : V \to \mathbb{Z} : \varphi|_{\partial V} = 0} \prod_{\{x,y\} \in E} e^{-U(\varphi_x - \varphi_y)}$$

Examples: ⊛ The $\mathbb{Z}$GF is a $\mathbb{Z}$UF

w/ $\mu_{\frac{1}{2}\lambda|\cdot|^2} = \delta_\lambda$.

⊛ The $\mathbb{Z}$BF (dual of XY model):

modified Bessel f^n.

$$e^{-U(q)} := I_q(\beta)$$

is a $\mathbb{Z}$UF.

μ is related to the cond. dist. of the norm of 2D BM.

⊛ $U_\alpha(q) := \lambda|q|^\alpha$ $\quad \alpha \in (0,2)$

is a $\mathbb{Z}$UF, via Bernstein's Thm.

$\exists$ lattice version of the $\mathbb{Z}$UF!

Let $\mathcal{L}$ be a general lattice (subset of $\mathbb{R}^n$, $n \equiv |V|$).

$$Z^{\mathbb{Z}UF, G, \mathcal{L}} := \sum_{\psi \in \mathcal{L}} \prod_{\{x,y\} \in E} e^{-U(\psi_x - \psi_y)}$$

defines adjacency structure on $\mathcal{L}$.

Thm. (annealed sub-lattice monotonicity):

If $\mathcal{M} \subseteq \mathcal{L}$ then

$$\mathbb{M}^{\mathbb{Z}UF, \mathbb{G}, \mathcal{M}}[\vartheta] \leq \mathbb{M}^{\mathbb{Z}UF, \mathbb{G}, \mathcal{L}}[\vartheta]$$

$\forall \quad \vartheta \in \mathbb{R}^n$.

Proof: Use ordinary RSD sub-lattice and FKG prop. of coupling const.

Divisibility holds for $\mathbb{Z}BF$ too.

Postulate it for $\mathbb{Z}UF$.