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Many-body Fermion Dynamics in $\mathbb{R}^d$

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joint work with

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- The issue of contuinuity of the dynamics
- A class of Hamiltonians for interacting fermions
- Free fermion dynamics
- Propagation bounds with many-body interactions
- Existence of infinite-volume dynamics
- Fermions in two dimensions with a constant magnetic field

(Non-)continuity of the dynamics

In a many body system with short-range interactions in $\mathbb{R}^d$, $\|[H, A]\|$, for local A, is typically unbounded unless the density of particles in the support of A is bounded.

Stability of Matter results address this issue in the ground state and in thermal equilibrium.

Unitary dynamics is well-defined on Fock space (states near the vacuum).

For non-equilibrium dynamics, the problem needs another solution.

An alternative approach was given by Narnhofer and Thirring (1990)), with the goal of preserving Galilean invariance.

A class of Hamiltonians for interacting fermions

$$\begin{aligned} H_\Lambda^\sigma &= \int_{\mathbb{R}^d} dx\, \nabla a_x^* \nabla a_x + V(x) a_x^* a_x \\ &\quad + \frac{1}{2} \int_\Lambda \int_\Lambda dxdy\;\, W(x-y) a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma) a(\varphi_y^\sigma) a(\varphi_x^\sigma). \end{aligned}$$

where V is an external potential such as a smooth periodic function, and W is a short-range two-body interaction.

We introduce a UV regularization in the interactions term using L^1-normalized Gaussians centered at $x \in \mathbb{R}^d$, and width $\sigma > 0$.

The standard pair interaction between point particles is recovered in the limit $\sigma \to 0$ which, for or N fermions, $N \geq 2$, converges in the strong resolvent sense to

$$H_N = \sum_{k=1}^{N} (-\Delta_k + V(x_k)) + \sum_{1 \leq k < l \leq N} W(x_k - x_l).$$

Free fermion dynamics

CAR($L^2(R^d)$) is generated by creation and annihilation operators $a^*(f)$, and $a(f)$, acting on Fock space, which satisfy $\{a(g), a^*(f)\} = \langle g, f\rangle \mathbb{1}$, and $\{a(g), a(f)\} = 0$.

For $W = 0$, there is a well-defined dynamics on CAR($L^2(R^d)$) corresponding to the one-particle Hamiltonians $h = -\Delta + V$. $u_t = e^{-ith}$ is a strongly continuous group of unitary operators on $L^2(R^d)$).

The second quantized dynamics is given by

$$\tau_t(a^*(f)) = a^*(u_t f), \quad \tau_t(a(f)) = a(u_t f), \quad f \in L^2(\mathbb{R}^d).$$

Using

$$\|\tau_t(a^*(f)) - \tau_s(a^*(f))\| = \|a^*((u_t - u_s)f)\| = \|(u_t - u_s)f\|,$$

we see that τ_t is a strongly continuous group of automorphisms on CAR($L^2(R^d)$).

The strong continuity at the level of the algebra of observables is not expected to survive for interacting fermions (or bosons) in the continuum (Bratteli-Robinson, Sakai)

Useful continuity of the dynamics of the infinite system is only expected in nice (physical) representations.

Propagation bounds with many-body interactions

Consider a pair-interaction given by a potential $W \in L^\infty(\mathbb{R}^d)$. A convenient way to write the interaction in second quantization is to use operator-valued distributions a_x, a_x^*, satisfying $\{a_x^*, a_y\} = \delta(x-y)\mathbb{1}$. Then, on a finite volume $\Lambda \subset \mathbb{R}^d$, one has

$$\frac{1}{2}\int_\Lambda \int_\Lambda dxdy\ W(x-y) a_x^* a_y^* a_y a_x = \bigoplus_{N=0}^{\infty} \sum_{1 \le k < l \le N} \chi_\Lambda(x)\chi_\Lambda(y) W(x_k - x_l).$$

For $W \neq 0$, this operator is not bounded.

UV regularization by replacing a_x by $a(\varphi_x^\sigma)$, with

$$\varphi_x^\sigma(y) = \frac{1}{(2\pi\sigma^2)^{d/2}} e^{-\frac{|y-x|^2}{2\sigma^2}} \quad \text{for all } y \in \mathbb{R}^d\,.$$

Then

$$\left\| \int_\Lambda \int_\Lambda dxdy\ W(x-y) a^*(\varphi_x^\sigma) a^*(\varphi_y^\sigma) a(\varphi_y^\sigma) a(\varphi_x^\sigma) \right\| \le \left(\frac{1}{4\pi\sigma^2} \right)^d \|W\|_\infty |\Lambda|^2.$$

It is straightforward to define a dynamics τ_t^Λ for the infinite system on $\mathbb{R}^d$ as the perturbation of the free dynamics $\tau_t^\emptyset$ by

$$W_\Lambda^\sigma = \frac{1}{2}\int_\Lambda \int_\Lambda dxdy \; W(x-y)a^*(\varphi_x^\sigma)a^*(\varphi_y^\sigma)a(\varphi_y^\sigma)a(\varphi_x^\sigma)$$

In particular, we have the Duhamel formula:

$$\tau_t^\Lambda(a(f)) = \tau_t^\emptyset(a(f)) + i\int_0^t ds\, \tau_s^\Lambda\left(\left[W_\Lambda^\sigma, \tau_{t-s}^\emptyset(a(f))\right]\right).$$

Since, $\tau_t^\emptyset$ is explicitly know, we focus on the difference with the perturbed dynamics. It turns out to be convenient to estimate the sum of the norms of the two basic anticommutators:

$$F_t^\Lambda(f,g) = \|\{\tau_t^\Lambda(a(f)), a^*(g)\} - \{\tau_t^\emptyset(a(f)), a^*(g)\}\| + \|\{\tau_t^\Lambda(a(f)), a(g)\}\|$$

Assumptions:

1. V is the Fourier transform of a finite signed measure with compact support. This includes periodic and quasi-periodic potentials of the form $a_1\cos(\omega_1\cdot x)+\cdots+a_n\cos(\omega_n\cdot x)$.

2. W is real, symmetric, and there are positive constants a and c_W for which

$$|W(x)|\leq c_W e^{-a|x|}.$$

Theorem (Lieb-Robinson bound, Gebert-N-Reschke-Sims, AHP 2020**)**

For any $\sigma>0$, V and W satisfying the assumptions above, and for all $t\in\mathbb{R}$ and any bounded, measurable set $\Lambda\subset\mathbb{R}^d$, we have for all $f,g\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$, the bound

$$F_t^\Lambda(f,g)\leq D(t)(e^{P_3(t)}-1)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}dxdy\, e^{-\frac{c_t|x-y|}{4}}|f(x)||g(y)|,$$

with functions $D(t)\sim e^{c|t||\ln|t||}$, P_3 a polynomial of degree $6d+1$ in $|t|$, and $c_t\sim\frac{1}{1+t^2}$.

Explicit expressions for the functions $D(t),P(t)$ and c_t can be given.

Existence of infinite-volume dynamics

Theorem (Existence of Thermodynamic Limit, Gebert-N-Reschke-Sims, AHP 2020)

Under the same assumptions, there exists a strongly continuous one-parameter group of automorphisms of the $CAR(L^2(\mathbb{R}^d))$, $\{\tau_t\}_{t\in\mathbb{R}}$, *such that for all* $f \in L^2(\mathbb{R}^d)$ *and any increasing sequence* (Λ_n) *of bounded measurable subsets of* $\mathbb{R}^d$ *such that* $\cup_n \Lambda_n = \mathbb{R}^d$,

$$\lim_{n\to\infty} \tau_t^{\Lambda_n}(a(f)) = \tau_t(a(f)),$$

in the operator norm topology, with convergence uniform in t in compact subsets of $\mathbb{R}$.

The proof of the Lieb-Robinson bound type uses the following propagation bound for the single-particle dynamics.

Theorem (Lieb-Robinson bound for Schrödinger operators, Gebert-N-Reschke-Sims, AHP 2020)

Let V be as above and consider the Schrödinger operator $H_1 = -\Delta + V$. Then there exist constants $C_1, C_2, C_3 > 0$ depending on d, μ, and σ such that the estimate

$$\left|\langle e^{-itH_1} f, \varphi_x^\sigma \rangle\right| \leq C_1 e^{C_2 |t| \ln |t|} \int_{\mathbb{R}^d} dy\, e^{-\frac{C_3}{t^2+1}|x-y|} |f(y)|$$

holds for all $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^d)$.

Note that the decay in space depends crucially on the smoothness of φ_x^σ. For example,

$$|(e^{-it(-\Delta)}\chi_{[-1,1]})(x)| \sim 2\sqrt{\frac{t}{\pi}} \frac{x}{x^2-1} \left|\sin \frac{x}{2t}\right|.$$

Fermions in $\mathbb{R}^2$ with a constant magnetic field

The one-particle dynamics is now given by the Landau Hamiltonian on $\mathbb{R}^2$;

$$h^{\mathrm{Landau}} = (\boldsymbol{p} - \boldsymbol{A})^2, \text{ with } \boldsymbol{A} = \frac{B}{2}(-x_2, x_1, 0).$$

where we chose the symmetric gauge for the magnetic field.

We make a corresponding change in the UV regularization by replacing the Gaussians ϕ_x^σ, which is a centered Gaussian translated by $x \in \mathbb{R}^2$, with a magnetically translated Gaussian:

$$\phi_x^\sigma(y) \to \psi_x^\sigma(y) = \frac{e^{-iy\cdot\boldsymbol{A}}}{2\pi\sigma^2} e^{-\frac{(x-y)^2}{2\sigma^2}}.$$

To simplify some estimates we make one more change: as before the pair potential functioned is assumed to be real, symmetric and bound but we assume Gaussian decay: there are constants a and c_W for which

$$|W(x)| \leq c_W e^{-a|x|^2}.$$

(there is a result for decay bounded by an exponential too).

The resulting Hamiltonian is

$$H_\Lambda^\sigma = d\Gamma(h^{\text{Landau}}) + \frac{1}{2}\int_\Lambda\int_\Lambda dxdy\ W(x-y)a^*(\psi_x^\sigma)a^*(\psi_y^\sigma)a(\psi_y^\sigma)a(\psi_x^\sigma).$$

Again, for fixed particle number the limit $\sigma \to 0$ exists in the strong resolvent sense.
Recall

$$F_t^\Lambda(f,g) = \|\{\tau_t^\Lambda(a(f)), a^*(g)\} - \{\tau_t^\emptyset(a(f)), a^*(g)\}\| + \|\{\tau_t^\Lambda(a(f)), a(g)\}\|.$$

Theorem (Lieb-Robinson Bound, Hingorani-N, in prep.)

For any $\sigma > 0$, and any bounded, measurable set $\Lambda \in \mathbb{R}^2$, let τ_t^Λ describe the Heisenberg dynamics associated with H_Λ^σ as above. Then, for all $f, g \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, we have constants A, α, and κ ssuch that

$$F_t^\Lambda(f,g) \leq A(e^{\kappa t} - 1)\int_{\mathbb{R}^4} dxdy\ |f(x)||g(y)|e^{-\frac{\alpha|x-y|^2}{\kappa}}$$

This is more than sufficient to prove the existence of a strongly continuous infinite system dynamics.

Note that the behavior in t is a simple exponential, as it is for lattice systems. This is due to fully localized nature of the free dynamics.

The spatial part decays as a Gaussian for pair potentials bounded by a Gaussian: one gets a 'diffusive' effect in addition to the non-interacting dynamics, which is localized on the scale $t \sim B$.

Theorem (Propagation Estimate for Landau Hamiltonian, Hingorani-N, in prep.)

Let Ψ_x^σ be the L^1 normalized magnetically translated Gaussian centered at x with width σ. Suppose $\sigma < \sqrt{2}\ell = \sqrt{\frac{2}{B}}$. for all $f \in L^2(\mathbb{R}^2)$ and $t \in \mathbb{R}$:

$$\left|\langle e^{-itH_L} f, \Psi_x^\sigma \rangle\right| \leq \frac{1}{2\pi\sigma^2} \int_{\mathbb{R}^2} dy \, |f(y)| e^{-\frac{\sigma^2 |x-y|^2}{8\ell^4}}.$$

Remarks

- It may not matter that for interacting pointlike Fermions in the continuum the dynamics is not continuous on the level of the algebra of observables, since no known physical effect depends on the exact pointlike nature.
- For fermionic atoms (i.e. not point particles), we expect a strongly continuous dynamics.
- A localizing effect of a magnetic field in two dimensions persists in the presence of interactions.