Interacting loop ensembles and Bose gases

Vedran Sohinger (University of Warwick)

Based on joint work with

Jürg Fröhlich (ETH Zürich)
Antti Knowles (University of Geneva)
Benjamin Schlein (University of Zürich)

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Outline

1. General setup of loop ensembles.
2. Specific examples: the Symanzik and Ginibre loop ensembles.
3. Setup of many-body quantum theory.
4. Statement of our results.

Loop ensembles

Let Ω = *loop space*.

Definition (Loop ensemble)

A *loop ensemble* Φ is a random point process on Ω.
$\rightarrow$ a random, locally finite collection of elements of Ω.

$\mathbb{L}$ = *single-loop measure*. $\rightarrow$ a (finite) positive measure on Ω.

Definition (Noninteracting loop ensemble)

The *noninteracting loop ensemble associated with* $\mathbb{L}$ is the Poisson point process on Ω with intensity measure $\mathbb{L}$.
$\rightarrow$ a random, locally finite collection of elements of Ω.

The loop configuration $\omega_1, \dots, \omega_n$ carries the weight

$$\frac{1}{Z}\,\frac{1}{n!}\,\mathbb{L}(\mathrm{d}\omega_1)\cdots\mathbb{L}(\mathrm{d}\omega_n)\,, \qquad Z = \sum_{n\in\mathbb{N}} \frac{1}{n!}\int \mathbb{L}(\mathrm{d}\omega_1)\cdots\mathbb{L}(\mathrm{d}\omega_n)\,.$$

Z-normalisation, $n!$-comes from permuting the ω_j.

Interacting loop ensembles

- *Two-loop interaction* is $\mathcal{V} : \Omega \times \Omega \to \mathbb{R}$.
- n*-loop interaction potential* is $V(\omega_1, \ldots, \omega_n) := \frac{1}{2} \sum_{i,j=1}^{n} \mathcal{V}(\omega_i, \omega_j)$.
- *Boltzmann factor* : $\mathrm{e}^{-V(\omega_1, \ldots, \omega_n)}$

Definition (Interacting loop ensemble)

Interacting loop ensemble Φ: Loop configuration $\omega_1, \ldots, \omega_n$ carries weight

$$\frac{1}{Z} \frac{1}{n!} \mathbb{L}(\mathrm{d}\omega_1) \cdots \mathbb{L}(\mathrm{d}\omega_n) \, \mathrm{e}^{-V(\omega_1, \ldots, \omega_n)} \,,$$

$$Z = \sum_{n \in \mathbb{N}} \frac{1}{n!} \int \mathbb{L}(\mathrm{d}\omega_1) \cdots \mathbb{L}(\mathrm{d}\omega_n) \, \mathrm{e}^{-V(\omega_1, \ldots, \omega_n)} \,.$$

p-loop correlations

Characterise Φ by *p-loop correlations* γ_p, $p \in \mathbb{N}^*$.
For all $f \geqslant 0$ symmetric

$$\int f(\omega_1, \dots, \omega_p)\, \gamma_p(\omega_1, \dots, \omega_p)\, \mathbb{L}(\mathrm{d}\omega_1) \cdots \mathbb{L}(\mathrm{d}\omega_p) = \mathbb{E}\left[\sum_{\omega_1, \dots, \omega_p}^{*} f(\omega_1, \dots, \omega_p)\right],$$

$$\sum_{\omega_1, \dots, \omega_p}^{*} \quad = \quad \text{sum over distinct } p\text{-tuples of loops.}$$

We obtain $\gamma_p(\omega_1, \dots, \omega_p) = \frac{Z(\omega_1, \dots, \omega_p)}{Z}$, where

$$Z(\omega_1, \dots, \omega_p) := \sum_{n \in \mathbb{N}} \frac{1}{n!} \int \mathbb{L}(\mathrm{d}\tilde{\omega}_1) \cdots \mathbb{L}(\mathrm{d}\tilde{\omega}_n)\, \mathrm{e}^{-V(\omega_1, \dots, \omega_p, \tilde{\omega}_1, \dots, \tilde{\omega}_n)},$$

$$Z = Z(\emptyset).$$

$\rightarrow$ For the noninteracting loop ensemble $\gamma_p = 1 \;\; \forall p \in \mathbb{N}^*$.

Notation

- Fix $d, L \in \mathbb{N}^*$ and work on the finite lattice $\Lambda := [-L/2, L/2)^d \cap \mathbb{Z}^d$.
- Discrete Laplacian: for $f : \Lambda \to \mathbb{C}$ consider

$$\Delta f(x) := \sum_{y:|y-x|=1} (f(y) - f(x)) \quad \text{(with periodic b.c.)}$$

- $\Omega^T_{y,x}$-set of $\omega : [0, T] \to \Lambda$ with $\omega(0) = x$, $\omega(T) = y$.
  (assume ω is 'càdlàg': right-continuous with left limits).
- Define

$$\Omega^T := \bigcup_{x,y \in \Lambda} \Omega^T_{y,x}\,, \qquad \Omega := \bigcup_{T \geqslant 0} \Omega^T\,.$$

Notation

- $\mathbb{P}_x^T$ = the law on Ω^T of the continuous-time random walk starting at x (Markovian jump process with generator $\Delta/2$). On Ω_T define

$$\mathbb{W}_{y,x}^T(\mathrm{d}\omega) := \mathbf{1}_{\omega(T)=y}\, \mathbb{P}_x^T(\mathrm{d}\omega)\,, \qquad \mathbb{W}^T(\mathrm{d}\omega) := \int_\Lambda \mathrm{d}x\, \mathbb{W}_{x,x}^T(\mathrm{d}\omega)\,.$$

- Let $\psi^t = \mathrm{e}^{t\Delta/2}$. For $f : \Lambda^n \to \mathbb{C}$, $0 < t_1 < \cdots < t_n < T$

$$\int \mathbb{W}_{y,x}^T(\mathrm{d}\omega)\, f(\omega(t_1), \dots, \omega(t_n)) = \int_{\Lambda^n} \mathrm{d}x_1 \cdots \mathrm{d}x_n\, \psi^{t_1}(x_1 - x)$$
$$\times\, \psi^{t_2 - t_1}(x_2 - x_1) \cdots \psi^{t_n - t_{n-1}}(x_n - x_{n-1})\, \psi^{T - t_n}(y - x_n)\, f(x_1, \dots, x_n)\,.$$

- $\mathbb{W}^T$: path measure for *closed paths* (loops).
- $\mathbb{W}_{y,x}^T$: path measure for *open paths* from x to y.

Example 1: The Symanzik loop ensemble

The ***Symanzik loop ensemble*** has single-loop measure

$$\mathbb{L}^{\mathrm{cl}}(\mathrm{d}\omega) := \int_0^\infty \mathrm{d}T\, \frac{\mathrm{e}^{-\kappa T}}{T}\, \mathbb{W}^T(\mathrm{d}\omega)\,.$$

- $\kappa > 0$ (the negative chemical potential)
  $\mathrm{e}^{-\kappa T}$: suppresses long loops.
- $1/T$-compensate overcounting for choice of origin in $[0, T]$
- $\mathbb{L}^{\mathrm{cl}}(\mathrm{d}\omega)$ is not finite (from small T contribution).
  $\rightarrow$ Consider an appropriate regularisation.
- *Two-loop interaction:* Given a two-body interaction potential $v : \Lambda \rightarrow \mathbb{R}$, let

$$\mathcal{V}^{\mathrm{cl}}(\omega, \tilde{\omega}) := \int_0^{T(\omega)} \mathrm{d}t \int_0^{T(\tilde{\omega})} \mathrm{d}\tilde{t}\, v\big(\omega(t) - \tilde{\omega}(\tilde{t})\big)\,.$$

  For $\omega \in \Omega^T_{y,x}$, write $T(\omega) = T$.
- *n-loop interaction potential:* $V^{\mathrm{cl}}(\omega_1, \ldots, \omega_n) := \frac{1}{2} \sum_{i,j=1}^n \mathcal{V}^{\mathrm{cl}}(\omega_i, \omega_j)$.

Interacting Euclidean field theories

The loop ensemble was introduced by Symanzik (1968) to describe *interacting Euclidean field theories*.

- $\phi : \Lambda \to \mathbb{C}$ distributed according to complex Gaussian measure with mean zero and covariance $(-\Delta + \kappa/2)^{-1}$

$$\mu_{(-\Delta+\kappa/2)^{-1}}(\mathrm{d}\phi) := \frac{1}{\pi^{|\Lambda|} \det(-\Delta/2+\kappa)^{-1}} \, \mathrm{e}^{\langle \phi, (\Delta/2-\kappa)\phi \rangle} \, .$$

- Relative (classical) partition function

$$\mathcal{Z}^{\mathrm{cl}} := \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi) \, \mathrm{e}^{-\frac{1}{2} \int_\Lambda \mathrm{d}x \int_\Lambda \mathrm{d}y \, |\phi(x)|^2 \, v(x-y) \, |\phi(y)|^2} \, .$$

- Classical p-point correlation function

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \frac{1}{\mathcal{Z}^{\mathrm{cl}}} \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi) \prod_{i=1}^{p} \bar{\phi}(y_i) \prod_{i=1}^{p} \phi(x_i) \\ \times \mathrm{e}^{-\frac{1}{2} \int_\Lambda \mathrm{d}x \int_\Lambda \mathrm{d}y \, |\phi(x)|^2 \, v(x-y) \, |\phi(y)|^2} \, .$$

Interacting Euclidean field theories

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \frac{1}{\mathcal{Z}^{\mathrm{cl}}} \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi) \prod_{i=1}^{p} \bar{\phi}(y_i) \prod_{i=1}^{p} \phi(x_i)$$
$$\times \mathrm{e}^{-\frac{1}{2} \int_\Lambda \mathrm{d}x \int_\Lambda \mathrm{d}y \, |\phi(x)|^2 \, v(x-y) \, |\phi(y)|^2} .$$

Symanzik's observation: We can write

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \sum_{\pi \in S_p} \left(\prod_{i=1}^{p} \int_0^{T_i} \mathrm{d}T_i \, \mathrm{e}^{-\kappa T_i} \right) \left(\prod_{i=1}^{p} \int \mathbb{W}^{T}_{y_{\pi(i)},x_i}(\mathrm{d}\omega_i) \right) \gamma_p^{\mathrm{cl}}(\omega_1, \ldots, \omega_p) \,,$$

$\gamma_p^{\mathrm{cl}}(\omega_1, \ldots, \omega_p)$: p-loop correlation functions of the loop ensemble.
$\to$ Consider p open paths $\omega_1, \ldots, \omega_p$.

Example 2: The Ginibre loop ensemble

Fix parameters $\nu, \kappa, \lambda > 0$. The ***Ginibre loop ensemble*** has single-loop measure

$$\mathbb{L}^{\nu,\kappa}(\mathrm{d}\omega) := \nu \sum_{T \in \nu \mathbb{N}^*} \frac{\mathrm{e}^{-\kappa T}}{T} \mathbb{W}^T(\mathrm{d}\omega) \,.$$

$\to$ Riemann sum approximation of $\mathbb{L}(\mathrm{d}\omega) = \int_0^\infty \frac{\mathrm{e}^{-\kappa T}}{T} \mathbb{W}^T(\mathrm{d}\omega)$.

- *Two-loop interaction:*

$$\mathcal{V}^{\nu,\lambda}(\omega, \tilde{\omega}) := \frac{\lambda}{\nu^2} \nu \sum_{s \in \nu \mathbb{N}} \mathbf{1}_{s < T(\omega)} \nu \sum_{\tilde{s} \in \nu \mathbb{N}} \mathbf{1}_{\tilde{s} < T(\tilde{\omega})} \times \frac{1}{\nu} \int_0^\nu \mathrm{d}t \, v\big(\omega(s+t) - \tilde{\omega}(\tilde{s}+t)\big) \,.$$

- For $\lambda = \nu^2$, view as discretisation of

$$\mathcal{V}^{\mathrm{cl}}(\omega, \tilde{\omega}) = \int_0^{T(\omega)} \mathrm{d}t \int_0^{T(\tilde{\omega})} \mathrm{d}\tilde{t} \, v\big(\omega(t) - \tilde{\omega}(\tilde{t})\big) \,.$$

Comparing the Symanzik and Ginibre loop ensemble

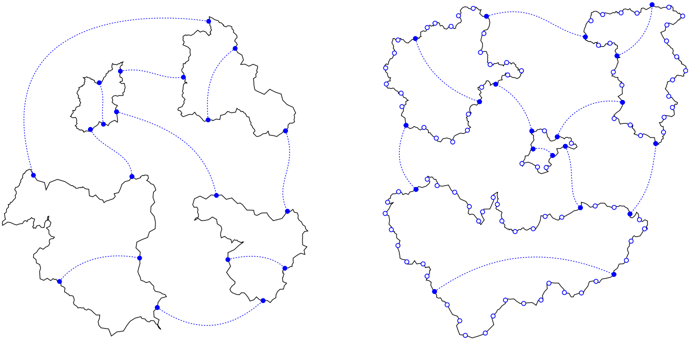

FIGURE. Left: Symanzik loop ensemble. Right: Ginibre loop ensemble.
Full lines: random loops, Dotted lines: interaction $\mathcal{V}(\omega, \tilde{\omega})$.

The interacting Bose gas at positive temperature

The loop ensemble was introduced by Ginibre (1964) to describe *interacting Bose gases at positive temperatures*.

- System of n spinless bosons of mass $m > 0$ confined to Λ, governed by the Hamiltonian

$$\mathbb{H}_n := -\sum_{i=1}^{n} \frac{\Delta_i}{2m} + \frac{\lambda}{2} \sum_{i,j=1}^{n} v(x_i - x_j)$$

  acting on $\mathcal{H}_n = L^2_{\mathrm{sym}}(\Lambda^n)$.

- *Grand canonical ensemble* at positive temperature. Equilibrium state described by sequence $(\rho_n)_n$

$$\rho_n := \frac{1}{\Xi} \mathrm{e}^{-\beta(\mathbb{H}_n - \mu n)}, \qquad \Xi := \sum_{n \in \mathbb{N}} \mathrm{Tr}\big(\mathrm{e}^{-\beta(\mathbb{H}_n - \mu n)}\big).$$

- $\beta > 0$: inverse temperature.
- $\mu < 0$: chemical potential.

The interacting Bose gas at positive temperature

Set $\beta = 1$. Replace m, μ with

$$\nu := \frac{1}{m} > 0\,, \qquad \kappa := -\mu m > 0\,.$$

Rewrite grand canonical ensemble as

$$\rho_n^{\nu,\kappa,\lambda} := \frac{1}{\Xi^{\nu,\kappa,\lambda}} \mathrm{e}^{-(H_n^{\nu,\lambda} + \kappa\nu n)}\,, \qquad \Xi^{\nu,\kappa,\lambda} := \sum_{n\in\mathbb{N}} \mathrm{Tr}\big(\mathrm{e}^{-(H_n^{\nu,\lambda}+\kappa\nu n)}\big)\,,$$

and Hamiltonian as

$$H_n^{\nu,\lambda} := -\frac{\nu}{2}\sum_{i=1}^n \Delta_i + \frac{\lambda}{2}\sum_{i,j=1}^n v(x_i - x_j)\,.$$

The *reduced p-particle density matrix* of the grand canonical ensemble is

$$\Gamma_p^{\nu,\kappa,\lambda} = \sum_{n\in\mathbb{N}} \frac{(p+n)!}{n!} \mathrm{Tr}_{p+1,\dots,p+n}\big(\rho_{p+n}^{\nu,\kappa,\lambda}\big)\,.$$

$\mathrm{Tr}_{p+1,\dots,p+n}$: the partial trace in $x_{p+1},\dots,x_{p+n}$.

Comparing the Ginibre and Symanzik representation

$(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}}$: operator kernel of $\Gamma_p^{\nu,\kappa,\lambda}$.
$\gamma_p^{\nu,\kappa,\lambda}(\omega_1,\ldots,\omega_p)$: correlation function of Ginibre loop ensemble.
Ginibre representation: Ginibre (1964)

$$\nu^p\,(\Gamma_p^{\nu,\kappa,\lambda})_{\mathbf{x},\mathbf{y}} = \sum_{\pi\in S_p}\left(\prod_{i=1}^p \nu \sum_{T_i\in\nu\mathbb{N}^*} \mathrm{e}^{-\kappa T_i}\right)\left(\prod_{i=1}^p \int \mathbb{W}^{T_i}_{y_{\pi(i)},x_i}(\mathrm{d}\omega_i)\right)\gamma_p^{\nu,\kappa,\lambda}(\omega_1,\ldots,\omega_p)\,.$$

$\rightarrow$ Compare with the *Symanzik representation*

$$(\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}} = \sum_{\pi\in S_p}\left(\prod_{i=1}^p \int_0^{T_i} \mathrm{d}T_i\,\mathrm{e}^{-\kappa T_i}\right)\left(\prod_{i=1}^p \int \mathbb{W}^{T}_{y_{\pi(i)},x_i}(\mathrm{d}\omega_i)\right)\gamma_p^{\mathrm{cl}}(\omega_1,\ldots,\omega_p)\,,$$

$\rightarrow$ Reduce to comparing $\gamma_p^{\nu,\kappa,\lambda}(\omega_1,\ldots,\omega_p)$ and $\gamma_p^{\mathrm{cl}}(\omega_1,\ldots,\omega_p)$.

The mean-field (classical field) limit

Mean-field limit: $\nu \to 0, \lambda = \nu^2$ and $\kappa > 0$ is fixed. Recall

$$\nu = \frac{1}{m} > 0\,, \qquad \kappa = -\mu m > 0\,. \quad (*)$$

$\to$ Interpret as large mass $+$ large chemical potential limit or high temperature $+$ high density limit.

Theorem 1: Fröhlich, Knowles, Schlein, S. (Preprint, 2020).

Let $v : \Lambda \to \mathbb{R}$ be pointwise nonnegative and of positive type ($\hat{v} \geqslant 0$ pointwise). Then the following limits hold.

(i) $\lim_{\nu \to 0} \mathcal{Z}^{\nu,\kappa,\nu^2} = \mathcal{Z}^{\mathrm{cl}}$.

(ii) $\lim_{\nu \to 0} \nu^p \, (\Gamma_p^{\nu,\kappa,\nu^2})_{\mathbf{x},\mathbf{y}} = (\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}}$ for all $p \in \mathbb{N}^*$ and $\mathbf{x}, \mathbf{y} \in \Lambda^p$.

$\to$ Previously shown by other methods Knowles (PhD Thesis, 2009).

The large-mass (classical particle) limit

Large-mass limit: $\nu \to 0, \lambda = 1, \kappa = \frac{\kappa_0}{\nu} > 0$ for fixed $\kappa_0 > 0$.

$$\nu = \frac{1}{m} > 0\,, \qquad \kappa = -\mu m > 0\,. \quad (*)$$

$\to$ Interpret as large mass limit for fixed chemical potential.

Theorem 2: Fröhlich, Knowles, Schlein, S. (Preprint, 2020).

Let $v : \Lambda \to \mathbb{R}$ be pointwise nonnegative . Then the following limits hold.

(i) $\lim_{\nu \to 0} \mathcal{Z}^{\nu,\kappa_0/\nu,1} = \mathcal{Z}^{\mathrm{lm}}$.

(ii) $\lim_{\nu \to 0} \nu^p \, (\Gamma_p^{\nu,\kappa_0/\nu,1})_{\mathbf{x},\mathbf{y}} = (\Gamma_p^{\mathrm{cl}})_{\mathbf{x},\mathbf{y}}$ for all $p \in \mathbb{N}^*$ and $\mathbf{x}, \mathbf{y} \in \Lambda^p$.

The results extend to the presence of a *hard core*, i.e. $R > 0$ s.t. $v(x) = \infty$ for $|x| \leqslant R$.

The large-mass (classical particle) limit

We describe $(\Gamma_p^{\mathrm{lm}})_{\mathbf{x},\mathbf{y}}$.
Interpretation: Ensemble of interacting stationary loops of integer time length.
Single-loop measure:

$$\mathbb{D}^{\mathrm{lm}}(\mathrm{d}\omega) = \sum_{k\in\mathbb{N}^*} \frac{\mathrm{e}^{-\kappa_0 k}}{k} \int_\Lambda \mathrm{d}x\, \mathbb{D}_x^k(\mathrm{d}\omega)$$

where $\mathbb{D}_x^T$ = atomic measure on Ω^T at the constant loop $\omega(t) = x$.
Two-loop interaction:

$$\mathcal{V}^{\mathrm{lm}}(\omega,\tilde{\omega}) := \sum_{0\leqslant k<T(\omega)} \sum_{0\leqslant \tilde{k}<T(\tilde{\omega})} \int_0^1 \mathrm{d}t\, v\big(\omega(k+t) - \tilde{\omega}(\tilde{k}+t)\big)\,.$$

$$(\Gamma_p^{\mathrm{lm}})_{\mathbf{x},\mathbf{y}} = \left(\prod_{i=1}^p \sum_{k_i\in\mathbb{N}^*} \mathrm{e}^{-\kappa_0 k_i}\, \delta(y_{\pi(i)} - x_i) \int \mathbb{D}_{x_i}^{k_i}(\mathrm{d}\omega_i)\right) \gamma_p^{\mathrm{lm}}(\omega_1,\dots,\omega_p)\,.$$

$\to$ **Paths collapse to points.**

The infinite-volume limit

Recall $\Lambda \equiv \Lambda_L = [-L/2, L/2)^d \cap \mathbb{Z}^d$. We are interested in taking $L \to \infty$. Fix $v \in \ell^1(\mathbb{Z}^d)$ nonnegative and of positive type. Consider

$$v^L : \Lambda_L \to \mathbb{R}\,, \qquad v^L(x) := \sum_{k \in (L\mathbb{Z})^d} v(x+k)\,.$$

Specific (relative) Gibbs potential of the Bose gas: $g^{\nu,\kappa,\lambda,L} := \frac{1}{|\Lambda_L|} \log \mathcal{Z}^{\nu,\kappa,\lambda,L}$.
Classical specific (relative) Gibbs potential: $g^{\mathrm{cl},L} := \frac{1}{|\Lambda_L|} \log \mathcal{Z}^{\mathrm{cl},L}$.

Theorem 3: Fröhlich, Knowles, Schlein, S. (Preprint, 2020).

Suppose that $\|v\|_{\ell^1}$ is sufficiently small. Then the following limits hold.

(i) $\lim_{\nu \to 0} \lim_{L \to \infty} g^{\nu,\kappa,\nu^2,L} = \lim_{L \to \infty} g^{\mathrm{cl},L}$.

(ii) $\lim_{\nu \to 0} \lim_{L \to \infty} \nu^p \left(\Gamma_p^{\nu,\kappa,\nu^2,L}\right) = \lim_{L \to \infty} \Gamma_p^{\mathrm{cl},L}$ for all $p \in \mathbb{N}^*$.

We prove an analogous result in the infinite-volume large-mass limit.

Overview of the literature

- Mean-field limit in the continuum for $d \leqslant 3$.
  - Lewin-Nam-Rougerie (J. É. Polytéchnique 2015, JMP 2018, Inventiones 2020).
  - Fröhlich-Knowles-Schlein-S. (CMP 2017, AIM 2019, JSP 2020, JAMS 2021).
  - S. (Preprint 2019, to appear in IMRN).
  - Rout-S. (Preprint 2022).

  On the lattice, we do not need to Wick order for $d = 2, 3$.
- Salmhofer (CMP 2021): Regularised coherent state functional integrals on the lattice.
- Random walk representation. In classical spin systems, Brydges-Fröhlich-Spencer (CMP 1982), Brydges-Fröhlich-Sokal (CMP 1983).
- Classical Gibbs state/Gibbs measure for the nonlinear Schrödinger equation and its invariance: Bourgain (CMP 1994), ...

Derivation of the Symanzik representation

For fixed $\varepsilon > 0$, we consider the regularised single-loop measure

$$\mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega) := \int_{\varepsilon}^{\infty} \mathrm{d}T \, \frac{\mathrm{e}^{-\kappa T}}{T} \, \mathbb{W}^{T}(\mathrm{d}\omega) \,.$$

$\to$ A finite measure. Recall that

$$V^{\mathrm{cl}}(\vec{\omega}) = \frac{1}{2} \sum_{i,j=1}^{n} \mathcal{V}^{\mathrm{cl}}(\omega_i, \omega_j) \,, \quad \vec{\omega} = (\omega_1, \ldots, \omega_n) \,,$$

where $\mathcal{V}^{\mathrm{cl}}(\omega, \tilde{\omega}) := \int_0^{T(\omega)} \mathrm{d}t \, \int_0^{T(\tilde{\omega})} \mathrm{d}\tilde{t} \, v\big(\omega(t) - \tilde{\omega}(\tilde{t})\big)$

$$\mathcal{Z}^{\mathrm{cl},\varepsilon} := \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_1) \cdots \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_n) \exp\big(-V^{\mathrm{cl}}(\vec{\omega})\big) \exp\big(K^{\varepsilon}\big) \,,$$

where

$$K^{\varepsilon} := -\int_{\varepsilon}^{\infty} \frac{\mathrm{d}T}{T} \, \mathrm{e}^{-\kappa T} \int \mathbb{W}^{T}(\mathrm{d}\omega) \,.$$

Claim: $\mathcal{Z}^{\mathrm{cl}} = \lim_{\varepsilon \to 0} \mathcal{Z}^{\mathrm{cl},\varepsilon}$.

Derivation of the Symanzik representation

- Fix $v : \Lambda \to \mathbb{R}$ of positive type ($\hat{v} \geqslant 0$).
- Obtain *positive quadratic form* $f \mapsto \langle f, vf \rangle \equiv \int \mathrm{d}x \int \mathrm{d}y f(x)\, v(x-y)\, f(y)$.
- $\mu_v =$ Gaussian measure on $\mathbb{R}^\Lambda$ with covariance v.

$$\int \mu_v(\mathrm{d}\sigma)\, \sigma(x)\, \sigma(y) = v(x-y)\,.$$

Hubbard-Stratonovich transformation. For $f : \Lambda \to \mathbb{R}$, we have

$$e^{-\frac{1}{2}\langle f, vf \rangle} = \int \mu_v(\mathrm{d}\sigma)\, \mathrm{e}^{\mathrm{i}\langle f, \sigma \rangle}\,.$$

The Feynman-Kac formula. For $w : \Lambda \to \mathbb{C}, t > 0$, we have

$$\left(\mathrm{e}^{t(\Delta/2 - w)}\right)_{y,x} = \int \mathbb{W}^t_{y,x}(\mathrm{d}\omega)\, \mathrm{e}^{-\int_0^t \mathrm{d}s\, w(\omega(s))}\,.$$

Derivation of the Symanzik representation

Apply Hubbard-Stratonovich transformation with $f = |\phi|^2$ to write

$$\mathcal{Z}^{\mathrm{cl}} = \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi)\, \mathrm{e}^{-\frac{1}{2} \int_\Lambda \mathrm{d}x \int_\Lambda \mathrm{d}y\, |\phi(x)|^2\, v(x-y)\, |\phi(y)|^2}$$
$$= \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi) \left(\int \mu_v(\mathrm{d}\sigma)\, \mathrm{e}^{\mathrm{i} \int \mathrm{d}x\, \sigma(x) |\phi(x)|^2} \right).$$

Use Fubini's theorem and evaluate a Gaussian integral:

$$\mathcal{Z}^{\mathrm{cl}} = \int \mu_v(\mathrm{d}\sigma) \int \mu_{(-\Delta/2+\kappa)^{-1}}(\mathrm{d}\phi)\, \mathrm{e}^{\mathrm{i} \int \mathrm{d}x\, \sigma(x) |\phi(x)|^2}$$
$$= \int \mu_v(\mathrm{d}\sigma) \det\big(-\Delta/2 + \kappa - \mathrm{i}\sigma\big)^{-1} \det\big(-\Delta/2 + \kappa\big).$$

Note that, for a fixed field σ (by $\det A = \exp\{\operatorname{Tr}\log A\}$)

$$\det\big(-\Delta/2 + \kappa - \mathrm{i}\sigma\big)^{-1} \det\big(-\Delta/2 + \kappa\big)$$
$$= \exp\Big\{ -\operatorname{Tr}\log\big(-\Delta/2 + \kappa - \mathrm{i}\sigma\big) + \operatorname{Tr}\log\big(-\Delta/2 + \kappa\big) \Big\}.$$

Derivation of the Symanzik representation

For $a, b \in \mathbb{C}, \operatorname{Re} a, \operatorname{Re} b > 0$, we have $\log a - \log b = -\int_0^\infty \frac{\mathrm{d}t}{t}\left(\mathrm{e}^{-ta} - \mathrm{e}^{-tb}\right)$. Using this identity and the Feynman-Kac formula, we get

$$-\operatorname{Tr}\log\bigl(-\Delta/2 + \kappa - \mathrm{i}\sigma\bigr) + \operatorname{Tr}\log\bigl(-\Delta/2 + \kappa\bigr) = \underbrace{\int_0^\infty \frac{\mathrm{d}T}{T} \int \mathbb{W}^T(\mathrm{d}\omega)\,\mathrm{e}^{-\kappa T}\left(\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))} - 1\right)}_{(*)}.$$

Recall $K^\varepsilon = -\int_\varepsilon^\infty \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-\kappa T} \int \mathbb{W}^T(\mathrm{d}\omega)$ and write $(*)$ as

$$\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega)\,\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))} + K^\varepsilon + \underbrace{\int_0^\varepsilon \frac{\mathrm{d}T}{T}\,\mathrm{e}^{-\kappa T} \int \mathbb{W}^T(\mathrm{d}\omega)\left(\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))} - 1\right)}_{\text{Small as } \varepsilon \to 0.}.$$

Obtain $\mathcal{Z}^{\mathrm{cl}} = \lim_{\varepsilon \to 0} \tilde{\mathcal{Z}}^{\mathrm{cl},\varepsilon}$, where

$$\tilde{\mathcal{Z}}^{\mathrm{cl},\varepsilon} := \int \mu_v(\mathrm{d}\sigma)\,\exp\left\{\int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega)\,\mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\,\sigma(\omega(t))} + K^\varepsilon\right\}.$$

Derivation of the Symanzik representation

Expand the exponential

$$\tilde{\mathcal{Z}}^{\mathrm{cl},\varepsilon} = \int \mu_v(\mathrm{d}\sigma)\, \exp\bigg\{ \int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega)\, \mathrm{e}^{\mathrm{i}\int_0^T \mathrm{d}t\, \sigma(\omega(t))} + K^\varepsilon \bigg\}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_1) \cdots \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_n) \left(\int \mu_v(\mathrm{d}\sigma)\, \mathrm{e}^{\mathrm{i}\sum_{i=1}^n \int_0^{T_i} \mathrm{d}t\, \sigma(\omega_i(t))} \right) \exp(K^\varepsilon)\,.$$

Apply the Hubbard-Stratonovich transformation with

$$f(x) = \sum_{i=1}^{n} \int_0^{T_i} \mathrm{d}t\, \delta(x - \omega_i(t))$$

($\rightarrow$ In this case $\sum_{i=1}^n \int_0^{T_i} \mathrm{d}t\, \sigma(\omega_i(t)) = \langle f, \sigma \rangle$.)
Deduce that

$$\tilde{\mathcal{Z}}^{\mathrm{cl},\varepsilon} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_1) \cdots \mathbb{L}^{\mathrm{cl},\varepsilon}(\mathrm{d}\omega_n)\, \exp\big(-V^{\mathrm{cl}}(\vec{\omega})\big) \exp\big(K^\varepsilon\big) = \mathcal{Z}^{\mathrm{cl},\varepsilon}\,.$$

Conclusion: $\mathcal{Z}^{\mathrm{cl}} = \lim_{\varepsilon \to 0} \mathcal{Z}^{\mathrm{cl},\varepsilon}$.$\square$

Thank you for your attention!