The spin 1 bilinear-biquadratic Heisenberg model on the complete graph

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Overview

1. Classical model

2. Quantum model

3. Ground state phase diagram in 3d

4. Ground state and finite temperature phase diagrams on the complete graph

Classical Heisenberg model

$$H = H_G(\sigma) = -\sum_{x,y} \sigma_x \cdot \sigma_y$$

x, y neighbouring vertices in finite graph G

$\sigma_x \in \mathbb{S}^2$

$$\text{Measure with density :} \quad \frac{1}{Z} e^{-\beta H}$$

$Z = \int_{(\mathbb{S}^2)^G} e^{-\beta H(\sigma)} d\sigma$

Classical bilinear-biquadratic model

$$H = H_{G,\phi} = -\sum_{x,y} \cos(\phi)\, \sigma_x \cdot \sigma_y + \sin(\phi)\, (\sigma_x \cdot \sigma_y)^2 \tag{1}$$

x, y neighbours in G; $\sigma_x \in \mathbb{S}^2$; measure density $\frac{1}{Z} e^{-\beta H}$

$\phi \in [0, 2\pi)$

Quantum Heisenberg model

$$H = H_G = -\sum_{x,y} \boxed{\mathbf{S}_x \cdot \mathbf{S}_y}$$

x, y neighbours in G

H an explicit matrix, acts on Hilbert space $\bigotimes_{x \in G} \mathbb{C}^{2S+1}$

S $\in \frac{1}{2}\mathbb{N}$ the spin number

$$\frac{1}{Z} e^{-\beta H}, \qquad Z = Z(\beta, G) = \mathrm{Tr} e^{-\beta H}$$

Quantum bilinear-biquadratic model

$$H = H_{G,\phi} = -\sum_{x,y} \cos(\phi)\, \mathbf{S}_x \cdot \mathbf{S}_y + \sin(\phi)\, (\mathbf{S}_x \cdot \mathbf{S}_y)^2$$

$\phi \in [0, 2\pi)$; H acts on $\bigotimes_{x \in G} \mathbb{C}^{2S+1}$

Set $S = 1$: Most general $SU(2)$-invariant system

Ground state phase diagram in $\mathbb{Z}^3$

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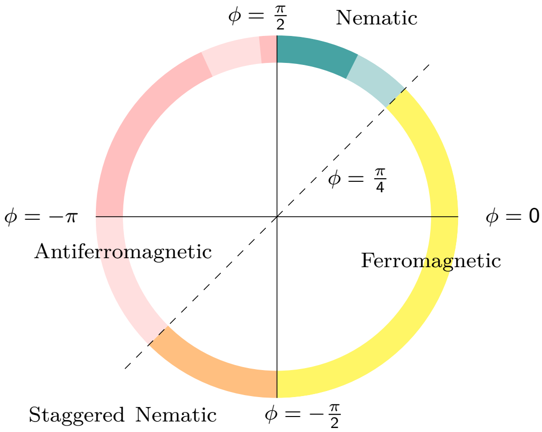

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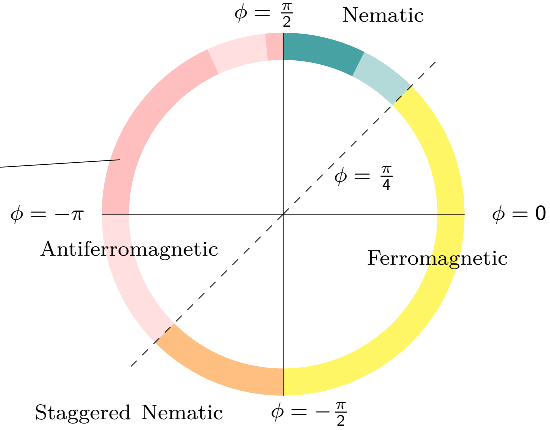
Néel, $\phi \in (2, \pi]$

DLS 1978; *Lees* 2016



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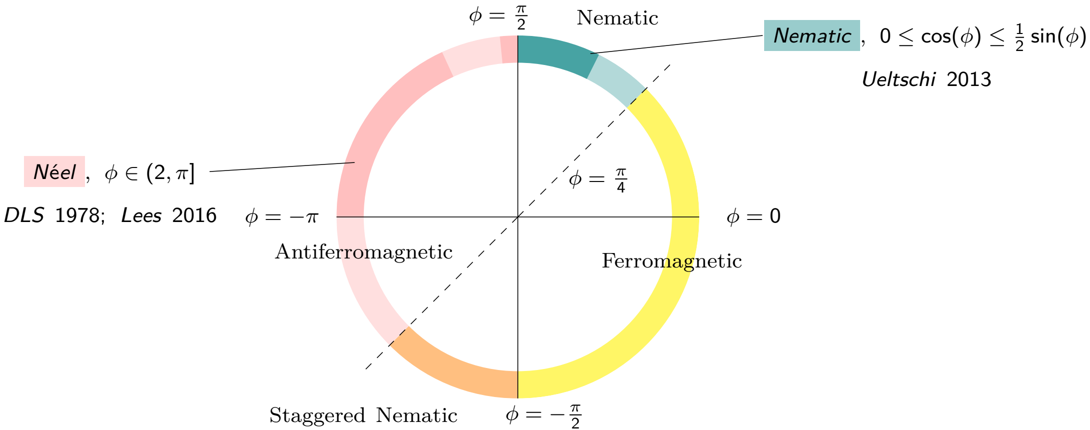

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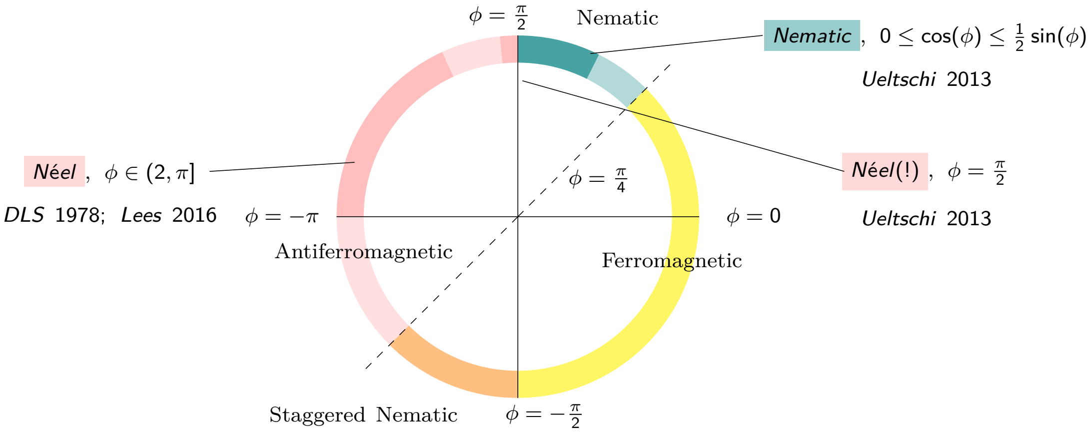

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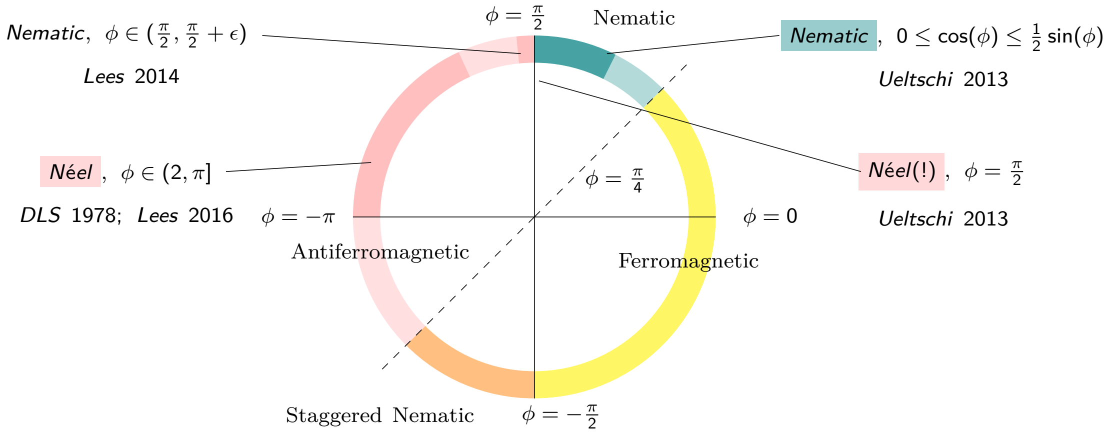

Method + history

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Random walk on S_n or Brauer algebra: Alon-Kozma, Berestycki-Kozma, others

Complete graph results

$$H = H_{G,\phi} = -\sum_{x,y} \cos(\phi)\, \mathbf{S}_x \cdot \mathbf{S}_y + \sin(\phi)\, (\mathbf{S}_x \cdot \mathbf{S}_y)^2; \qquad G = K_n$$

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Ground state and finite temperature phase diagrams; critical temperatures

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$$\Phi(\beta,\phi,h) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{Tr}\left[\exp\left(-\frac{\beta}{n} H + h \sum_x S_x^{(i)}\right)\right]$$

$$m = m(\beta,\phi) = \frac{\partial \Phi}{\partial h}\Big|_{h=0}$$

Ground state phase diagram on the complete graph

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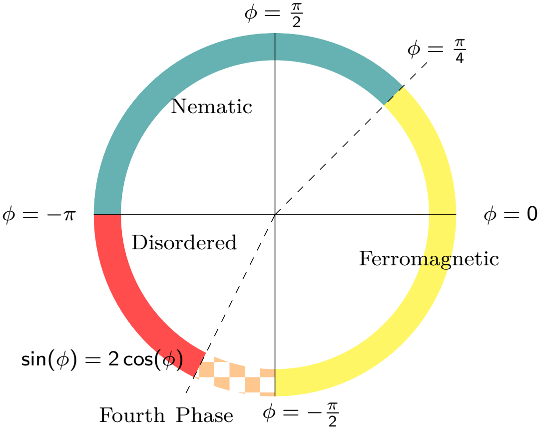

Ground states: $\mathbb{Z}^3$ vs the complete graph

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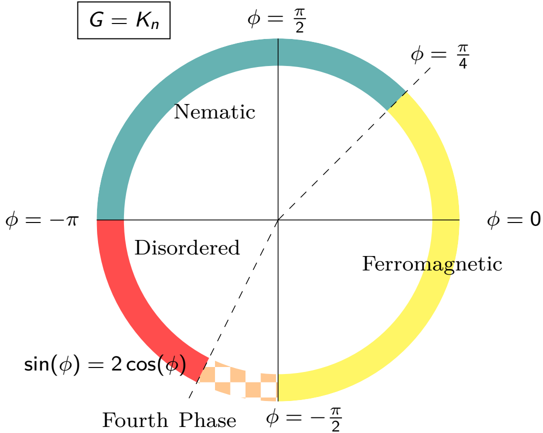

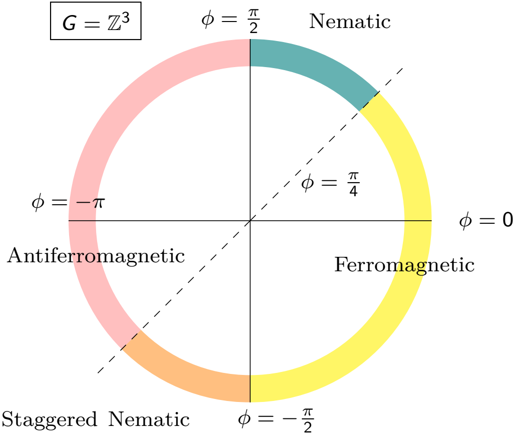

Ground state and finite temperature diagrams

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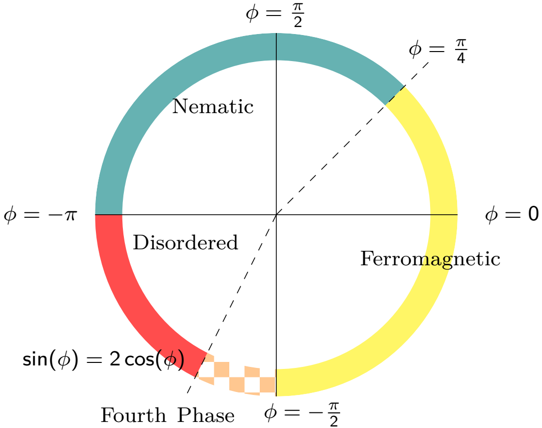

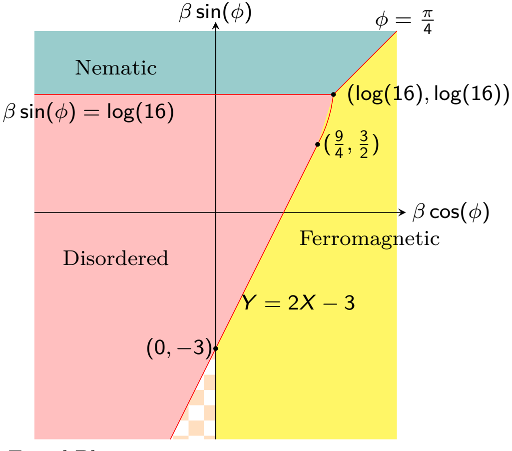

On the complete graph

$$H = H_{G,\phi} = -\sum_{x,y} \cos(\phi)\, \mathbf{S}_x \cdot \mathbf{S}_y + \sin(\phi)\, (\mathbf{S}_x \cdot \mathbf{S}_y)^2$$

Theorem (R. 2020)

Let $G = K_n$.

$$\lim_{n\to\infty} \frac{1}{n} \log \operatorname{Tr}\left[e^{-\frac{\beta}{n}H}\right] = \max_{(x,y)\in\Delta} \frac{\beta}{2}\left(\sin(\phi)\sum_{i=1}^{3} x_i^2 + (cos(\phi) - \sin(\phi))y^2\right) - \sum_{i=1}^{3} x_i \log x_i$$

where $\Delta = \{(x_1, x_2, x_3, y) \in [0,1]^4 \;:\; \sum x_i = 1,\; x_i \geq x_{i+1},\; y \leq x_1 - x_3\}$