The excitation spectrum for two-dimensional bosons in the Gross-Pitaevskii regime

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based on joint works with S. Cenatiempo and B. Schlein

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Bose-Einstein Condensation

A *Bose-Einstein condensate* is a state of matter which occurs in dilute gases of bosonic atoms at very low-temperatures.
Roughly speaking $\rightsquigarrow$ a macroscopic fraction of the particles behaving as if they were occupying the same one-particle state.

Final goal: occurrence of BEC in the thermodynamic limit $\rightsquigarrow$ far away from rigorous results.

Mathematical results:

- 3D very well-known (BEC for MF and GP regimes, thermodynamic functions known in several regimes);
- 2D got attention later.

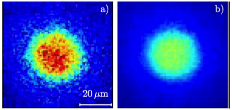

Figure: Absorption imaging of quasi-2D clouds of Rubidium-87 atoms. Yefsah et al. 2011

Why 2D?

- Physically relevant for applications;
- Theoretically: critical case for BEC $\rightsquigarrow$ Mermin-Wagner theorem prevents condensation at finite temperature.

Mathematical formulation of BEC

- N particle in $\mathbb{R}^2$, the state is described by the wave function $\psi_N \in L^2_s(\mathbb{R}^{2N})$, s.t. $\|\psi_N\|_2 = 1$. Energy of the system described by H_N;
- A mathematical formalization of BEC for general interacting system by Onsager-Penrose '56 in terms of one-particle reduced density matrix

$$\gamma_N^{(1)}(x;y) = \int_{\mathbb{R}^{2(N-1)}} \overline{\psi_N(x, x_2, ..., x_N)}\psi_N(y, x_2, ..., x_N) dx_2 \cdots dx_N$$

for $\psi_N \in L^2_s(\mathbb{R}^{2N})$, $\|\psi_N\|_2 = 1$, can be diagonalized $\gamma_N^{(1)} = \sum_k \lambda_k |\varphi_j\rangle\langle\varphi_j|$

BEC $\Leftrightarrow$ $\lambda_0 = \max_j \lambda_j$ is of order $\mathcal{O}(1)$, corresponding eigenvector is the condensate wave function φ_0.

Meaning: all particles, up to a fraction vanishing in the limit $N \to \infty$, are condensated in the one-particle state φ_0.

Thermodynamic limit vs scaling limits

- **Thermodynamic limit:** N interacting bosons confined in a box with area L^2 $\rightsquigarrow$ $N, L \to \infty$ with the density $\rho = N/L^2$ fixed;
- **Dilute regimes:** address the problem in simpler, but still physically relevant, settings, letting the interaction potential depends on the number of particles N, where N is large.

  The N-dependent potential $\rightsquigarrow$ effective description for interactions occurring in large many-particle systems.

Examples of effective theories:

- Hartree theory for weak interactions, i.e. range of the interaction potential much larger than interparticle distance;
- Gross-Pitaevskii theory for strongly interacting systems.

Known results 2D thermodynamic limit

For N bosons interacting through a fixed potential with scattering length $\mathfrak{a}$, confined in a box with area L^2, so that $N, L \to \infty$ with the density $\rho = N/L^2$ kept fixed, $b = |\log(\rho\mathfrak{a}^2)|^{-1}$, in the dilute limit $\rho\mathfrak{a}^2 \ll 1$, the g.s. energy per particle is given by

$$e_0(\rho) = 4\pi\rho^2 b\Big(1 + b\log b + \big(1/2 + 2\gamma + \log\pi\big)b + o(b)\Big)\,,$$

with γ the Euler's constant.

- Schick '71 predictions for the g.s. energy of 2D gases in the thermodynamic limit (leading order); confirmed by Lieb-Yngvason '01;
- second order predictions by Andersen '02, Pilati-Boronat-Casulleras-Giorgini '05, Mora-Castin '09;
- Second order recently proved by Fournais-Girardot-Junge-Morin-Olivieri '22 (previous results restricting to quasi-free states Fournais-Napiórkowski-Reuvers-Solovej '19).

2D bosons in Gross-Pitaevskii regime

N interacting bosons in a 2D unit box $\Lambda = [-1/2, 1/2]^2$. The energy of the system is described by the Hamiltonian acting on $L^2_s(\Lambda^N)$

$$H_N = \underbrace{\sum_{j=1}^{N} - \Delta_{x_j}}_{\mathcal{O}(N)} + \underbrace{\sum_{i<j}^{N} e^{2N} V(e^N(x_i - x_j))}_{\mathcal{O}(N^2)},$$

Remarks:

- Expectation of H_N on a factorized state $\varphi_0^{\otimes N}$ is of order $\mathcal{O}(N^2)$;
- The exponential scaling comes from the 2D scattering length $\mathfrak{a}$ of V

$$\frac{2\pi}{\log(R/\mathfrak{a})} = \inf_{\phi \in H^1(B_R)} \int_{B_R} \left[|\nabla \phi|^2 + \frac{1}{2} V |\phi|^2 \right] dx$$

for $R > R_0$, with R_0 range of the potential and with $\phi(x) = 1$ for all x with $|x| = R$.

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Scattering equation 2D Gross-Pitaevskii

The unique minimizer satisfies $-\Delta\phi^{(R)} + \frac{1}{2}V\phi^{(R)} = 0$

$$\phi^{(R)}(x) = \frac{\log(|x|/\mathfrak{a})}{\log(R/\mathfrak{a})} \quad \text{for } R_0 < |x| \leq R\,.$$

By scaling, $\phi_N(x) := \phi^{(e^N R)}(e^N x)$ is such that

$$-\Delta\phi_N + \frac{1}{2}e^{2N}V(e^N x)\phi_N = 0.$$

We have

$$\phi_N(x) = \frac{\log(|x|/\mathfrak{a}_N)}{\log(R/\mathfrak{a}_N)} \qquad \forall x \in \mathbb{R}^2 : e^{-N}R_0 < |x| \leq R\,,$$

for all $x \in \mathbb{R}^2$ with $e^{-N}R_0 < |x| \leq R$. Here $\mathfrak{a}_N = e^{-N}\mathfrak{a}$.
$\rightsquigarrow$ if we consider

$$\int_{B_R} e^{2N}V(e^N x)\phi_N(x) \sim \mathcal{O}\left(\tfrac{1}{N}\right)$$

Previous results

- Lieb-Seiringer-Yngvason '01,'06 , Lieb-Seiringer '01,'02 for bosons confined by external trapping potentials:
  - the ground state energy E_N of H_N is such that

$$E_N = 2\pi N\big(1 + O(N^{-1/5})\big)\,; \tag{1}$$

  - BEC in the zero-momentum mode $\varphi_0(x) = 1$ for all $x \in \Lambda$, for any approximate ground state $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and

$$\lim_{N\to\infty} \tfrac{1}{N}\langle \psi_N, H_N \psi_N\rangle = 2\pi, \tag{2}$$

  the one-particle reduced density matrix $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N}|\psi_N\rangle\langle\psi_N|$ is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)}\varphi_0\rangle \leq CN^{-\bar\delta} \quad \text{suff. small } \bar\delta > 0\,. \tag{3}$$

- Schnee -Yngvason '06 g.s. energy at leading order and BEC for 3d bosons in a trap strongly confined in one direction.

Almost optimal rate for BEC

Theorem (C.-Cenatiempo-Schlein '21)

Let $V \in L^3(\mathbb{R}^2)$, spherically symmetric, compactly supported and pointwise non-negative. Consider a sequence $\psi_N \in L^2_s(\Lambda^N)$ with $\|\psi_N\| = 1$ and s.t.

$$\langle \psi_N, H_N \psi_N \rangle \leq 2\pi N + K \qquad \text{for } K > 0.$$

Then the reduced density matrix $\gamma_N^{(1)} = \mathrm{tr}_{2,\dots,N} |\psi_N\rangle\langle\psi_N|$ is s.t.

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq \frac{C(1+K)}{N} \qquad N \in \mathbb{N} \text{ large enough.}$$

Remarks:

- We have bounds (first order) for g.s.e. $2\pi N - C \leq E_N \leq 2\pi N + C \log N$
- **expected** order of the correction for the lower bound, but **logarithmic correction** for the upper bound;
- convergence of 1-p.d. matrix expected to be **optimal**, but we need to choose $K = C \log N$ to have $\langle \psi_N, H_N \psi_N \rangle \leq 2\pi N + K$;
- the condition $V \in L^3(\mathbb{R}^2)$ comes from properties of scattering equation.

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The excitation spectrum & second order corrections

Theorem (C.-Cenatiempo-Schlein '22)

Let $V \in L^3(\mathbb{R}^2)$ as before. The g.s.e. E_N of H_N is such that, as $N \to \infty$,

$$E_N = 2\pi(N-1) + \pi \log(2\mathfrak{a}^2) + \frac{1}{2} \sum_{p \in 2\pi\mathbb{Z}^2 \setminus \{0\}} \left[\sqrt{p^4 + 8\pi p^2} - p^2 - 4\pi + \frac{(4\pi)^2}{2p^2} (1 - J_0(|p|/\sqrt{2})) \right] + \mathcal{O}(N^{-\frac{1}{10}+\delta})$$

for any $\delta > 0$. Moreover, the spectrum of $H_N - E_N$ below a threshold $\zeta > 0$ consists of eigenvalues having the form

$$\sum_{p \in 2\pi\mathbb{Z}^2 \setminus \{0\}} n_p \sqrt{p^4 + 8\pi p^2} + \mathcal{O}(N^{-\frac{1}{10}+\delta}(1+\zeta^{17})) \quad \forall \delta > 0\,.$$

Remarks:

- J_0 is the zero-th order Bessel function of the first kind
- $n_p \in \mathbb{N}$ for all $p \in 2\pi\mathbb{Z}^2 \setminus \{0\}$, $n_p \neq 0$ for finitely many $p \in 2\pi\mathbb{Z}^2 \setminus \{0\}$ only.

Highlights of the proof of BEC & Bogoliubov theory

- Use techniques developed in BOCCATO-BRENNECKE-CENATIEMPO-SCHLEIN '19-'20 for 3D bosons in Gross-Pitaevskii regime;
- work in the Fock space and through the action of unitary operators we renormalize the Hamiltonian H_N to get the correct energy expectation. How?

$$e^{-A}e^{-B}U_N H_N U_N^* e^B e^A = \mathcal{R}_N = \text{Const} + Q_{\mathcal{R}} + \mathcal{C}_{\mathcal{R}} + \mathcal{H}_N + \mathcal{E}_{\mathcal{R}}$$

- e^B unitary operator, B quadratic, whose coefficient takes into account correlation [ERDÖS-SCHLEIN-YAU '08, BENEDIKTER-DE OLIVEIRA-SCHLEIN '15]. It is defined through the modified scattering equation

$$\big(-\Delta + \tfrac{1}{2}V(x)\big) f_\ell(x) = \lambda_\ell f_\ell(x) \qquad |x| \le e^N \ell$$

  $f_\ell(x) = 1$ when $|x| = e^N\ell$, with $\ell = N^{-\alpha}$;
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Highlights of the proof of Bogoliubov theory

Using a-priori knowledge on the number of the excitations we can reduce the renormalized Hamiltonian

$$\mathcal{R}_N = \text{Const.} + \mathcal{Q}_{\mathcal{R}} + \mathcal{V}_N + \mathcal{E}_N$$

Upper bound

- $\mathcal{V}_N = \frac{1}{2} \sum_{\substack{p,q\in\Lambda_+^*, r\in\Lambda^*: \\ r\neq -p,-q}} \widehat{V}(r/e^N) a^*_{p+r} a^*_q a_p a_{q+r}$ is of order $\mathcal{O}(1)$, cannot be neglected;
- apply another renormalization $e^{-D}\mathcal{R}_N e^D$, quartic in terms of creation and annihilation operators $\rightsquigarrow$ reduction to $\mathcal{V}_N^{(H)}$ that can be controlled on low-excited states.
- e^{B_τ} Bogoliubov-type transformation, that diagonalizes $e^{-D}\mathcal{R}_N e^D$.

Highlights of the proof of Bogoliubov theory

Lower bound

- $\mathcal{V}_N$ can be neglected because it is non-negative $\rightsquigarrow \mathcal{R}_N \geq \text{Const.} + \mathcal{Q}_{\mathcal{R}} + \mathcal{E}_N$;
- e^{B_ν} Bogoliubov-type transformation to diagonalize [Seiringer '00]
- localization on errors of the form $\mathcal{N}^2/N$ (cannot obtain bounds for power of $\mathcal{N}$ due to large contribution from the cubic) [Nam-Triay '21, Hainzl-Schlein-Triay '22, (before Lewin-Nam-Serfaty-Solovej '15,)]

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The excitation Hamiltonian is defined as $\mathcal{L}_N := U_N H_N U_N^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, with

$$\begin{aligned}
\mathcal{L}_N^{(0)} &= \tfrac{1}{2}\widehat{V}(0)(N-1)(N-\mathcal{N}_+) + \tfrac{1}{2}\widehat{V}(0)\mathcal{N}_+(N-\mathcal{N}_+) \\
\mathcal{L}_N^{(2)} &= \mathcal{K} + N\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)a_p^*a_p\Big(1-\tfrac{\mathcal{N}_+}{N}\Big) + \tfrac{N}{2}\sum_{p\in\Lambda_+^*}\widehat{V}(p/e^N)\left[b_p^*b_{-p}^* + \mathrm{h.c.}\right] \\
\mathcal{L}_N^{(3)} &= \sqrt{N}\sum_{p,q\in\Lambda_+^*:p+q\neq 0}\widehat{V}(p/e^N)\left[b_{p+q}^*a_{-p}^*a_q + \mathrm{h.c.}\right] \\
\mathcal{L}_N^{(4)} &= \tfrac{1}{2}\sum_{\substack{p,q\in\Lambda_+^*,r\in\Lambda^*:\\ r\neq -p,-q}}\widehat{V}(r/e^N)a_{p+r}^*a_q^*a_pa_{q+r} = \mathcal{V}_N\,.
\end{aligned}$$

- One can notice that $\langle\Omega,\mathcal{L}_N\Omega\rangle$ is of order $\mathcal{O}(N^2)$.
- b_p^*, b_p are modified creation and annihilation operators, such that

$$U_N^*b_p^*U_N = a_p^*\frac{a_0}{\sqrt{N}}, \qquad U_N^*b_pU_N = \frac{a_0^*}{\sqrt{N}}a_p,$$

$\rightsquigarrow$ b_p^* creates a particle with momentum $p\in\Lambda_+^* = 2\pi\mathbb{Z}^2\setminus\{0\}$ and annihilates a particle from the condensate;

Quadratic and cubic renormalization

- We take into account correlations through properties of the scattering function;
- from f_ℓ we define the coefficients

$$\check{\eta}(x) = -N\big(1 - f_\ell(e^N x)\big) \quad \rightsquigarrow \quad \|\check{\eta}\|_{L^2}^2 = \|\eta\|_{L^2}^2 \leq C\ell^2\,;$$

- we choose $\ell = N^{-\alpha}$, so that the norm $\mathcal{O}(N^{-\alpha})$;
- with η_p, Fourier coefficients of $\check{\eta}(x)$, we introduce generalized Bogoliubov transformation through the anti-symmetric operator

$$B = \frac{1}{2}\sum_{p\in\Lambda_+^*}\left(\eta_p b_p^* b_{-p}^* - \bar{\eta}_p b_p b_{-p}\right)$$

  with $\eta_{-p} = \eta_p$ for all $p \in \Lambda_+^*$, b_p^*, b_p are modified creation and annihilation operators $\rightsquigarrow$ a_p, a_p^* do not ensure that the truncated Fock space is kept invariant [ESY '08, BDS '15, BS '19];
- cubic phase is defined as

$$A := \frac{1}{\sqrt{N}}\sum_{r,v\in\Lambda_+^*}\eta_r\left[b_{r+v}^* a_{-r}^* a_v - \text{h.c.}\right];$$

Quartic renormalization and final diagonalization

Upper bound

- quartic phase is defined as

$$D := \frac{1}{4N} \sum_{r \in \Lambda_+^*, v, w \in P_L} \eta_r \left[a^*_{r+v} a^*_{w-r} a_v a_w - \text{h.c.} \right];$$

- final last Bogoliubov-type transformation that rotates the g.s. vector

$$B_\tau = \frac{1}{2} \sum_{p \in P_L} \left(\tau_p b^*_p b^*_{-p} - \bar{\tau}_p b_p b_{-p} \right)$$

Lower bound

- Neglect $\mathcal{V}_N$ by positivity
- apply B_ν, like B_τ with a different kernel [Seiringer '00]
- localize on sectors of $\mathcal{N}^2/N$, assorbing errors in the kinetic energy operator (for a lower bound we have errors at most errors of this form).

Beyond condensation and Bogoliubov theory

- Norm-approximation for low-energy states $\rightsquigarrow$ and compute the condensate depletion;
- investigate fluctuations of observables measured on the ground state $\rightsquigarrow$ validity of a central limit theorem for one particle observables measured on the condensate Rademacher-Schlein '19, Rademacher '20 ;