The interchange model on two-block graphs

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The model

General form of Hamiltonian:

$$H_n = -\sum_{1 \le i < j \le n} \alpha_{i,j} T_{i,j}, \qquad \alpha_{i,j} \in \mathbb{R}$$

where $T_{i,j}$ = transposition of i and j tensor factors in $(\mathbb{C}^r)^{\otimes n}$

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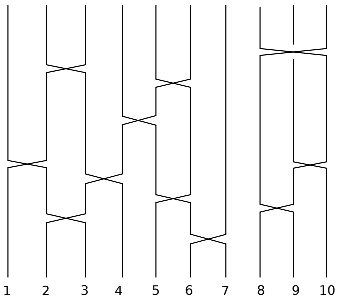
- Generally: $T_{i,j}$ = polynomial in $\vec{S}_i \cdot \vec{S}_j$

Probabilistic representation for ferromagnet

If all $\alpha_{i,j} \geq 0$ then $-H_n$ is a generator for a Markov chain:



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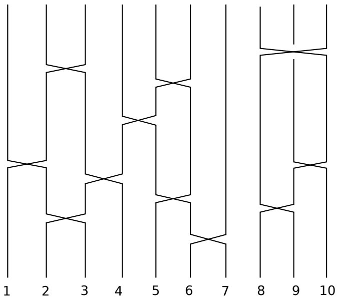

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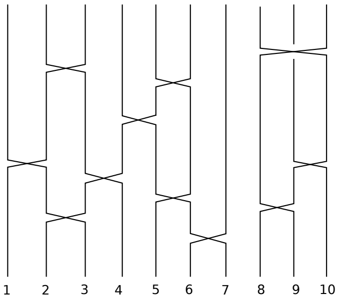

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$Z_n = \mathrm{tr}[e^{-\beta H_n}] = c\mathbb{E}[r^{\#\text{cycles}}]$

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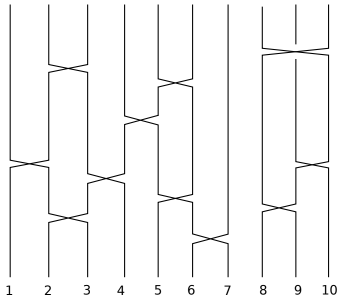

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Our choice of couplings:

$$\alpha_{i,j} = \frac{1}{n}\begin{cases} a, & i,j \in A = \{1,2,\dots,m\}, \\ b, & i,j \in B = \{m+1,\dots,n\}, \\ c, & i \in A, j \in B. \end{cases}$$

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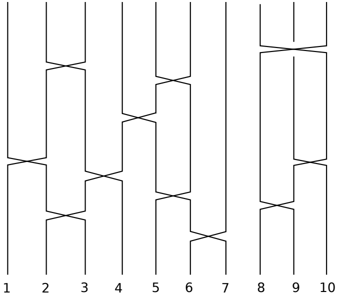

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If $a = b = c$: complete graph (B.–Fröhlich–Ueltschi '19)

Result I: free energy

From now on $a, b, c \in \mathbb{R}$ and

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As $n \to \infty$ with $m/n \to \rho \in (0,1)$,

$$\tfrac{1}{n} \log \operatorname{tr}[e^{-\beta H_n}] \to \max_{(x;y)\in\Omega_\rho} \big(\mathcal{H}(x;y) + \beta\mathcal{E}(x;y)\big) + cst$$

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$Q(\xi,\eta) = \frac{1}{2}(a\xi^2 + b\eta^2 + 2c\xi\eta)$

Result II: phase transition

Proposition

- *If* $Q \leq 0$ *(i.e.* $a, b \leq 0$*,* $ab \geq c^2$*) then for all* $\beta \geq 0$*, maximum attained only at* $\omega_0 = (\frac{\rho}{r}, \ldots, \frac{\rho}{r}; \frac{1-\rho}{r}, \ldots, \frac{1-\rho}{r})$

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For $r = 2$ (i.e. spin $\frac{1}{2}$):

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For $r \geq 3$ (i.e. spin ≥ 1) and a certain subset of parameters a, b, c:

- we have a formula for β_{c}
- the phase transition is discontinuous (phase coexistence at β_{c})

Result III: spin-density Laplace transform

Let $W = \mathrm{diag}(w_1, \ldots, w_r)$ where $w_1, \ldots, w_r \in \mathbb{C}$ and $W_i = \mathbb{I} \otimes \cdots \otimes W \otimes \cdots \otimes \mathbb{I}$.

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Interpretation: extremal Gibbs states indexed by the orbit of $X + Y$ under $\mathcal{U}(r)$.

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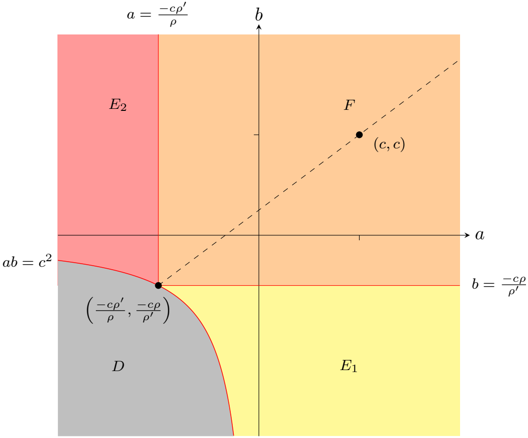
For $c > 0$:



F: ferromagnetic

D: disordered

E_1: ferromagnetic in block A, partly disordered in B

E_2: vice versa

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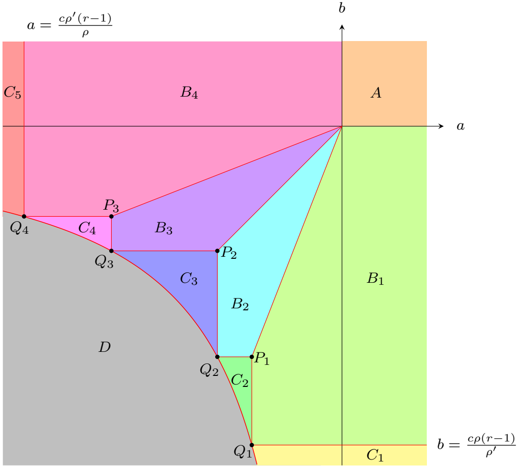
For $c < 0$, with $r = 5$ (i.e. spin 2):



A: antiferromagnetic

D: disordered

B_k: intermediate

C_k: partly disordered

Method: Schur–Weyl duality

Write

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$$(\mathbb{C}^r)^{\otimes n} \cong \bigoplus_{\lambda\vdash n,\ell(\lambda)\le r} s_\lambda V^\lambda$$

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Moreover, as an $S_m \times S_{n-m}$-representation

$$V^\lambda \cong \bigoplus_{\mu\vdash m, \nu\vdash n-m} c^\lambda_{\mu,\nu} V^\mu \otimes V^\nu$$

This decomposition diagonalizes the Hamiltonian (Schur's Lemma)

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Key input: "Horn's inequalities" (Knutson–Tao theorem) $c^{\lambda}_{\mu,\nu} > 0$ if and only if there are Hermitian $r \times r$ matrices X, Y with spectra μ, ν respectively, such that $X + Y$ has spectrum λ

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Then on $V^{\mu} \otimes V^{\nu}$,
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For spin-density Laplace transform, use full Schur–Weyl duality: as a representation of $\mathrm{GL_r}(\mathbb{C}) \times \mathrm{S_n}$

$$(\mathbb{C}^r)^{\otimes n} \cong \bigoplus_{\lambda \vdash n, \ell(\lambda) \leq r} U^{\lambda} \otimes V^{\lambda}$$

Thank you!