The Euclidean Φ^4_2 Theory as the Limit of an Interacting Bose Gas

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The nonlinear Schrödinger equation

Consider the spatial domain $\Lambda = \mathbb{T}^d$ for $d = 1, 2, 3$.

- Study the ***nonlinear Schrödinger equation (NLS)***.

$$\begin{cases} \mathrm{i}\partial_t \phi_t(x) = \big(-\Delta/2 + \kappa\big)\phi_t(x) + \int \mathrm{d}y\, w(x-y)\, |\phi_t(y)|^2\, \phi_t(x) \\ \phi_0(x) = \Phi(x) \in H^s(\Lambda)\,. \end{cases}$$

- Parameter $\kappa > 0$. Sobolev space $\|f\|_{H^s(\Lambda)} := \|(1 + |\xi|)^s\, \widehat{f}(\xi)\|_{L^2_\xi}$.
- Interaction potential
  1. **Nonlocal problem:**
     $w : \Lambda \to \mathbb{R}$ is even, integrable and of *positive type:* $\widehat{w} \geqslant 0$.
  2. **Local problem:** $w = \delta$.
- Conserved energy

$$s(\phi) = \int \mathrm{d}x\, \bar{\phi}(x)(\kappa - \Delta/2)\phi(x) + \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}y\, |\phi(x)|^2\, w(x-y)\, |\phi(y)|^2\,.$$

- Main object of study: $d = 2$ and $w = \delta$.
  $\to$ Gives rise to the (complex) Euclidean Φ^4_2 theory.

Gibbs measures for the NLS

- The ***Gibbs measure*** $\mathrm{d}\mu$ associated with s is the probability measure on the space of fields $\phi : \Lambda \to \mathbb{C}$

$$\mu(\mathrm{d}\phi) := \frac{1}{\zeta}\,\mathrm{e}^{-s(\phi)}\,\mathrm{d}\phi\,, \qquad \zeta := \int \mathrm{e}^{-s(\phi)}\,\mathrm{d}\phi\,.$$

  $\mathrm{d}\phi =$ (formally-defined) Lebesgue measure.

- A *field theory.*
  $\to$ for $w = \delta$: *(complex) Euclidean* Φ^4_d *theory*.
- Formally, $\mathrm{d}\mu$ is invariant under the flow of the NLS:

$$(\mathrm{F}_t)_* \mathrm{d}\mu = \mathrm{d}\mu\,,$$

  where F_t := flow map of NLS.

Known results

- **Rigorous construction of measure:** CQFT literature in the 1970-s (Nelson, Glimm-Jaffe, Simon).
- **Proof of invariance under flow of NLS:** Bourgain and Zhidkov (1990s).
  $\rightarrow$ Measure supported on low-regularity Sobolev spaces.
- **Application to nonlinear dispersive PDEs:** *Obtain low-regularity solutions of NLS μ-almost surely*.
  Recent advances: Bourgain-Bulut, Burq-Tzvetkov, Burq-Thomann-Tzvetkov, Cacciafesta- de Suzzoni, Deng-Nahmod-Yue, Fan-Ou-Staffilani-Wang, Genovese-Lucà-Valeri, Nahmod-Oh-Rey-Bellet-Staffilani, Nahmod-Rey-Bellet-Sheffield-Staffilani, Oh-Pocovnicu, Oh-Tzvetkov-Wang, Thomann-Tzvetkov, Tzvetkov, ...
- **Stochastic PDEs:** Stationary measure for a nonlinear heat equation driven by space-time white noise; Stochastic quantisation.
  Lebowitz-Rose-Speer, Nelson, Parisi-Wu,...
  Recent works: Da Prato-Debussche, Gubinelli-Imkeller-Perkowski, Hairer, Kupiainen,...

NLS as a classical limit

NLS is a classical limit of many-body quantum theory.

- On $\mathfrak{H}^{(n)} \equiv L^2_{\mathrm{sym}}(\Lambda^n)$ we consider the n*-body Hamiltonian*

$$H^{(n)} := -\frac{1}{2}\sum_{i=1}^{n} \Delta_i + \sum_{i,j=1}^{n} w_n(x_i - x_j)\,.$$

- Solve *n-body Schrödinger equation*

$$\mathrm{i}\partial_t \Psi_{n,t} = H^{(n)} \Psi_{n,t}\,.$$

  Obtain that, as $n \to \infty$

$$\Psi_{n,0} \sim \phi_0^{\otimes n} \quad \text{implies} \quad \Psi_{n,t} \sim \phi_t^{\otimes n}\,.$$

  (Hepp (1974), Ginibre-Velo (1979), Spohn (1980), Fröhlich-Tsai-Yau (1998), Fröhlich-Knowles-Pickl (2006), Erdős-Schlein-Yau (2006, 2007), Fröhlich-Graffi-Schwarz (2007), Fröhlich-Knowles-Schwarz (2009), T. Chen-Pavlović (2010), Pickl (2010), Ammari-Nier (2011), . . .).

- **Problem:** Obtain Gibbs measure $\mathrm{d}\mu$ as many-body quantum limit .

Gibbs measures for $d = 1$

- Let $s_0(\phi) := \int \mathrm{d}x \, (|\nabla \phi(x)|^2/2 + \kappa |\phi(x)|^2)$.
  Define the ***Wiener measure*** $\mathrm{d}\mu_0$

$$\mu_0(\mathrm{d}\phi) := \frac{1}{\zeta_0} \, \mathrm{e}^{-s_0(\phi)} \, \mathrm{d}\phi \,, \quad \zeta_0 := \int \mathrm{e}^{-s_0(\phi)} \, \mathrm{d}\phi \,.$$

- Typical elements in the support of $\mathrm{d}\mu_0$ have form

$$\sum_{k \in \mathbb{Z}^d} \frac{g_k(\omega)}{(|k|^2 + \kappa)^{1/2}} \, \mathrm{e}^{2\pi i k \cdot x} \,, \ (g_k) = \text{i.i.d. complex Gaussians.}$$

  $\to$ ***Classical free field***. Series converges almost surely in $H^{1-\frac{d}{2}-\varepsilon}(\Lambda)$.
- The *classical interaction* is

$$W := \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, |\phi(x)|^2 \, w(x-y) \, |\phi(y)|^2 \,.$$

- In $[0, +\infty)$ almost surely if $d = 1$ (with $w \in L^1$ or $w = \delta$).
- In this case $\mathrm{d}\mu$ is a well-defined probability measure on $H^{1/2-\varepsilon}(\mathbb{T}^1)$ which satisfies

$$\mathrm{d}\mu \ll \mathrm{d}\mu_0 \,.$$

Wick ordering

- For $d = 2, 3$, $W = \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y\, |\phi(x)|^2\, w(x-y)\, |\phi(y)|^2$ is *infinite almost surely* even if $w \in L^\infty(\mathbb{T}^d)$.
- Perform a *renormalisation* in the form of ***Wick ordering***.
  1. **Nonlocal problem** ($w \in L^1$)

$$W^w := \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y :|\phi(x)|^2: \; w(x-y) \; :|\phi(y)|^2:$$
$$= \frac{1}{2} \int \mathrm{d}x\, \mathrm{d}y \left(|\phi(x)|^2 - \mathbb{E}_{\mu_0}[|\phi(x)|^2]\right) w(x-y) \left(|\phi(y)|^2 - \mathbb{E}_{\mu_0}[|\phi(y)|^2]\right).$$

Note $W^w \geqslant 0$ if w is of positive type, by using that for f real

$$\int \mathrm{d}x\, \mathrm{d}y\, f(x) w(x-y) f(y) = \sum_k \hat{w}(k)\, |\hat{f}(k)|^2\,.$$

  2. **Local problem** ($w = \delta$)

$$W^w := \frac{1}{2} \int \mathrm{d}x \; :|\phi(x)|^4:$$
$$= \frac{1}{2} \int \mathrm{d}x \Big(|\phi(x)|^4 - \underbrace{4\mathbb{E}_{\mu_0}[|\phi(x)|^2]\, |\phi(x)|^2}_{\text{mass renormalisation}} + \underbrace{2\mathbb{E}_{\mu_0}[|\phi(x)|^2]^2}_{\text{energy renormalisation}}\Big)\,.$$

$\to$ Here, W^w is no longer nonnegative.

The Φ_d^4 measure

We fix $w = \delta$.

- When $d = 2$, we have $W \equiv W^w \in L^2(\mathrm{d}\mu_0)$, but W is unbounded below.
  Nevertheless, one has $\mathrm{e}^{-W(\phi)} \in L^1(\mathrm{d}\mu_0)$.
  $\rightarrow$ Shown by Nelson (1973).
- When $d = 3$, we have $W \notin L^2(\mathrm{d}\mu_0)$.
  $\rightarrow$ Construction requires a further mass renormalisation.
  Glimm-Jaffe, Feldman-Osterwalder, Park, Gawędzki-Kupiainen, Brydges-Dimock-Hurd, Brydges-Fröhlich-Sokal, Gubinelli-Hofmanová, Barashkov-Gubinelli, ...
- When $d \geqslant 4$, it is expected that μ is Gaussian no matter how one renormalises the interaction.
  Shown for $d \geqslant 5$ by Fröhlich (1982) and Aizenman (1982).
  Shown for $d = 4$ for a real field ϕ by Aizenman and Duminil-Copin (2020).

The classical Gibbs state

- Classical Gibbs state $\rho(\cdot)$: Given $X \equiv X(\omega)$ a random variable, let

$$\rho(X) := \frac{\int X \, \mathrm{e}^{-W} \, \mathrm{d}\mu_0}{\int \mathrm{e}^{-W} \, \mathrm{d}\mu_0} = \int X \, \mathrm{d}\mu \, .$$

On $\mathfrak{H}^{(p)} \equiv L^2_{\mathrm{sym}}(\Lambda^p)$ define the ***classical p-particle correlation function*** γ_p by its operator kernel

$$(\gamma_p)_{x_1,\dots,x_p;y_1,\dots,y_p} := \rho\big(\overline{\phi}(y_1)\cdots\overline{\phi}(y_p)\phi(x_1)\cdots\phi(x_p)\big) \, .$$

$\to$ μ is determined by $(\gamma_p)_p$.

The quantum problem

- Given $m > 0$ (mass of particles) and $\lambda > 0$ (coupling constant), we work with the Hamiltonian on $\mathfrak{H}^{(n)}$ given by

$$H^{(n)} := \frac{1}{m} \sum_{i=}^{n} \Big(-\frac{\Delta_i}{2} + \kappa \Big) + \frac{\lambda}{2} \sum_{i,j=1}^{n} w(x_i - x_j) - an + b \,.$$

$a, b \in \mathbb{R}$: Renormalisation parameters.

  - a: mass renormalisation.
  - b: energy renormalisation.

The quantum problem

- We henceforth write $\nu = \frac{1}{m}$ and take $\lambda = \nu^2$.
- The n-body Hamiltonian is

$$H_\nu^{(n)} := \nu \sum_{i=1}^{n} \Big(-\frac{\Delta_i}{2} + \kappa \Big) + \frac{\nu^2}{2} \sum_{i,j=1}^{n} w(x_i - x_j) - a_\nu n + b_\nu \,.$$

- The ***grand canonical ensemble*** is the sequence $(\rho_n)_n$ given by

$$\rho_n \equiv \rho_{\nu,n} := \frac{1}{Z_\nu} \mathrm{e}^{-H_\nu^{(n)}} \,, \qquad Z_\nu := \sum_{n \in \mathbb{N}} \mathrm{Tr}_{\mathfrak{H}^{(n)}} \mathrm{e}^{-H_\nu^{(n)}} \,.$$

- For $p \in \mathbb{N}$, one defines the ***p-particle reduced density matrix*** $\gamma_{\nu,p}$ by its operator kernel

$$\gamma_{\nu,p} := \sum_{n \geqslant p} \frac{n!}{(n-p)!} \, \mathrm{Tr}_{p+1,\dots,n}(\rho_n) \,.$$

Statement of the results: nonlocal problem, w continuous

Theorem 1: Fröhlich, Knowles, Schlein, S. (2020).

Suppose that w is *continuous* and of *positive type*.
Then, we have

$$\mathcal{Z}_\nu := \frac{Z_\nu}{Z_\nu^{(0)}} \to \zeta \quad \text{as} \quad \nu \to 0 \qquad (\mathrm{PF})\,.$$

Moreover, for all $p \in \mathbb{N}$ we have

$$\gamma_{\nu,p} \xrightarrow{L^r} \gamma_p \quad \text{as} \quad \nu \to 0 \qquad (\mathrm{CF})\,.$$

Here,

$$r \in \begin{cases} [1,\infty]\,, & d = 1 \\ [1,\infty)\,, & d = 2 \\ [1,3)\,, & d = 3\,. \end{cases}$$

is optimal. We apply Wick ordering when $d = 2, 3$.

Statement of the results: nonlocal problem, w continuous

Theorem 1 can be deduced from a stronger result.
We Wick order $\gamma_{\nu,p}$ and γ_p to obtain $\widehat{\gamma}_{\nu,p}$ and $\widehat{\gamma}_p$.
Example: We have

$$\widehat{\gamma}_1 = \gamma_1 - \gamma_1^{(0)}$$

and

$$(\widehat{\gamma}_2)_{x_1,x_2;\tilde{x}_1,\tilde{x}_2} = (\gamma_2)_{x_1,x_2;\tilde{x}_1,\tilde{x}_2} - (\gamma_1)_{x_1;\tilde{x}_1}\,(\gamma_1^{(0)})_{x_2;\tilde{x}_2} - (\gamma_1)_{x_1;\tilde{x}_2}\,(\gamma_1^{(0)})_{x_2;\tilde{x}_1} - (\gamma_1)_{x_2;\tilde{x}_1}\,(\gamma_1^{(0)})_{x_1;\tilde{x}_2} - (\gamma_1)_{x_2;\tilde{x}_2}\,(\gamma_1^{(0)})_{x_1;\tilde{x}_1} + (\gamma_2^{(0)})_{x_1,x_2;\tilde{x}_1,\tilde{x}_2}\,.$$

Theorem 2: Fröhlich, Knowles, Schlein, S. (2020).

For all $p \in \mathbb{N}$ we have

$$\widehat{\gamma}_{\nu,p} \xrightarrow{C} \widehat{\gamma}_p \quad \text{as} \quad \nu \to 0\,, \qquad (\mathrm{CF} - \mathrm{Wick})\,.$$

where $\xrightarrow{C}$ denotes convergence in the space of continuous functions w.r.t. $\|\cdot\|_{L^\infty}$.

Statement of the results: local problem, $d = 2$

Let $v : \mathbb{R}^2 \to \mathbb{R}$ be even, smooth, compactly supported, of positive type and integral 1. We let

$$w^\varepsilon(x) := \sum_{n \in \mathbb{Z}^2} \frac{1}{\varepsilon^2} v\Big(\frac{x-n}{\varepsilon}\Big).$$

Theorem 3: Fröhlich, Knowles, Schlein, S. (2022).

Fix $d = 2$. Suppose that $\varepsilon \equiv \varepsilon(\nu)$ satisfies

$$\varepsilon \geqslant \exp\left(-(\log \nu^{-1})^{1/2-c}\right)$$

for some $c > 0$.
Then (PF) and $(\mathrm{CF}-\mathrm{Wick})$ hold as $\varepsilon, \nu \to 0$ with classical interaction being

$$W = \frac{1}{2}\int \mathrm{d}x \, :|\phi(x)|^4: \,.$$

Statement of the results: nonlocal problem, $w \in L^q$

Consider $w \in L^q(\Lambda)$ even, real-valued, and of positive type, such that

$$\begin{cases} q > 1 & \text{if } d = 2 \\ q > 3 & \text{if } d = 3\,. \end{cases}$$

These are the optimal integrability conditions for the choice of w noted by Bourgain (1997). Consider

$$w^\varepsilon := v * \delta_\varepsilon \in C^\infty(\Lambda)\,, \qquad \delta_\varepsilon = \frac{1}{\varepsilon^d} \sum_{y \in \mathbb{Z}^d} F\Big(\frac{x-y}{\varepsilon}\Big)\,.$$

Theorem 4: Fröhlich, Knowles, Schlein, S. (2022).

Suppose that $\varepsilon \equiv \varepsilon(\nu)$ satisfies $\varepsilon(\nu) \gtrsim \frac{1}{\log \nu^{-1}}$. Then (PF) and $(\mathrm{CF-Wick})$ hold as $\varepsilon, \nu \to 0$ with classical interaction being

$$W = \frac{1}{2} \int \mathrm{d}x\,\mathrm{d}y\ :|\phi(x)|^2:\ w(x-y)\ :|\phi(y)|^2:\ .$$

Related results

- $1D$ results: previously shown using variational techniques by Lewin, Nam, Rougerie (2015).
  Higher dimensions: non local, non translation-invariant interactions.
- Fröhlich, Knowles, Schlein, S. (2017): analysis of translation-invariant interactions $w \in L^\infty$ for $d = 2, 3$ by using perturbative methods, with a modified Gibbs state. New proof of $d = 1$ result.
- S. (2019): Extension of above result to $w \in L^q$ for optimal q following Bourgain (1997).
- Lewin, Nam, Rougerie (2018): $1D$ non-periodic problem with subharmonic trapping.
- Lewin, Nam, Rougerie (2018): $2D$ problem with translation-invariant interaction for smooth w without modified Gibbs state.
- Lewin, Nam, Rougerie (2020): Extension to $3D$.
- Fröhlich, Knowles, Schlein, S. (2018): time-dependent problem in $1D$.
  $\rightarrow$ *Corresponds to the invariance of the measure*.
- Rout-S. (2022): $1D$ focusing problem.
- Fröhlich, Knowles, Schlein, S. (2020): Analysis of problem on the lattice using *loop ensembles*.

The Φ_2^4 theory. Proof of Theorem 3

We compare the fully Wick-ordered interaction

$$W = \frac{1}{2}\int \mathrm{d}x \ :|\phi(x)|^4:$$

with

$$W^\varepsilon = \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\tilde{x}\ :|\phi(x)|^2:\ w^\varepsilon(x-\tilde{x})\ :|\phi(\tilde{x})|^2: - \tau^\varepsilon \int \mathrm{d}x\ :|\phi(x)|^2: - E^\varepsilon$$

where

$$\tau^\varepsilon := \int \mathrm{d}x\, w^\varepsilon(x)\, G(x)\,, \qquad E^\varepsilon := \frac{1}{2}\int \mathrm{d}x\,\mathrm{d}\tilde{x}\, w^\varepsilon(x-\tilde{x})\, G(\tilde{x}-x)^2$$

and

$$G = (-\Delta/2+\kappa)^{-1}$$

is the Green function.

We analyse (PF).

- **Step 1:** Compare $\mathcal{Z}_\nu \equiv \mathcal{Z}_{\nu,\varepsilon}$ with $\zeta^\varepsilon \equiv \int \mathrm{e}^{-W^\varepsilon} \mathrm{d}\mu_0$ using a quantitative version of the bosonic functional integral from the proof of Theorem 1.
  - We show that
$$\left|\mathcal{Z}_\nu - \zeta^\varepsilon\right| \lesssim \mathrm{e}^{C(\log \varepsilon^{-1})^2} \, \nu^{1/4} \,.$$
  - Due to the Wick ordering of the full quartic nonlinearity, we destroy the positivity of the interaction.
    $\to W^\varepsilon$ is not necessarily positive.
  - This analysis needs to be quantitative.

The Φ_2^4 theory. Proof of Theorem 3

We analyse (PF).

- **Step 2:** Compare $\zeta^\varepsilon \equiv \int \mathrm{e}^{-W^\varepsilon} \mathrm{d}\mu_0$ and $\zeta \equiv \int \mathrm{e}^{-W} \mathrm{d}\mu_0$.
  $\to$ A purely field theoretic step.
  - Prove a version of the *Nelson argument (1973)* for nonlocal interactions.
  - Pass through an intermediate interaction

$$V^\varepsilon := \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}\tilde{x} \, w^\varepsilon(x - \tilde{x}) \ : |\phi(x)|^2 \, |\phi(\tilde{x})|^2 :$$

    and show that for all $t \geqslant 1$

$$\mu_0\big(\mathrm{e}^{-V^\varepsilon} > t\big) \lesssim \exp\big(\mathrm{e}^{-c\sqrt{\log t}}\big) \, .$$

  - Deduce that $\|\mathrm{e}^{-V^\varepsilon}\|_{L^p(\mu_0)}$ is uniformly bounded in ε.

The Φ_2^4 theory. Proof of Theorem 3

- **Step 2:** Compare $\zeta^\varepsilon \equiv \int \mathrm{e}^{-W^\varepsilon} \mathrm{d}\mu_0$ and $\zeta \equiv \int \mathrm{e}^{-W} \mathrm{d}\mu_0$.
  - Using Wick's theorem show that $\lim_{\varepsilon \to 0} \|W^\varepsilon - V^\varepsilon\|_{L^2(\mathrm{d}\mu_0)} = 0$ and $\lim_{\varepsilon \to 0} \|V^\varepsilon - W\|_{L^2(\mathrm{d}\mu_0)} = 0$.
  - Conclude using hypercontractivity.

$$\left\|\mathrm{e}^{V^\varepsilon - W^\varepsilon} - 1\right\|_{L^p(\mu_0)} \leqslant \sum_{k \geqslant 1} \frac{1}{k!} \|(V^\varepsilon - W^\varepsilon)^k\|_{L^p(\mu_0)}$$

$$= \sum_{k \geqslant 1} \frac{1}{k!} \|V^\varepsilon - W^\varepsilon\|^k_{L^{pk}(\mu_0)} \lesssim \sum_{k \geqslant 1} \frac{1}{k!} (pk)^k \|V^\varepsilon - W^\varepsilon\|^k_{L^2(\mu_0)}.$$

  - Deduce that $\|\mathrm{e}^{-W^\varepsilon}\|_{L^p(\mu_0)}$ is uniformly bounded in ε.
  - Conclude by

$$\left\|\mathrm{e}^{-W} - \mathrm{e}^{-W^\varepsilon}\right\|_{L^p(\mu_0)} \leqslant \int_0^1 \mathrm{d}t \left\|(W - W^\varepsilon)\, \mathrm{e}^{-tW^\varepsilon - (1-t)W}\right\|_{L^p(\mu_0)}.$$

Proof of Theorem 1: Functional integral (formal setup)

- Quantum field $\Phi : [0,\nu] \times \Lambda \to \mathbb{C}$, measure $\mathrm{D}\Phi := \prod_{\tau \in [0,\nu]} \prod_{x \in \Lambda} \Phi(\tau, x)$.
- Define

$$\mathbf{S^0}(\Phi) := \int_0^\nu \mathrm{d}\tau \int_\Lambda \mathrm{d}x\, \bar{\Phi}(\tau,x)(\partial_\tau + \kappa - \Delta/2)\Phi(\tau,x)$$

$$\mathbf{W}(\Phi) := \frac{1}{2}\int_0^\nu \mathrm{d}\tau \int_{\Lambda^2} \mathrm{d}x\,\mathrm{d}y\, |\Phi(\tau,x)|^2\, w(x-y)\, |\Phi(\tau,y)|^2$$

$$\mathbf{S}(\Phi) := \mathbf{S^0}(\Phi) + \mathbf{W}(\Phi)\,.$$

- Quantum (relative) partition function $\mathcal{Z}_\nu = \frac{\int \mathrm{D}\Phi\, \mathrm{e}^{-\mathbf{S}(\Phi)}}{\int \mathrm{D}\Phi\, \mathrm{e}^{-\mathbf{S^0}(\Phi)}}$ (formally).
- Rescale for $t \in [0,1]$ as $\Phi'(t,x) := \sqrt{\nu}\Phi(\nu t, x)$.

$$\mathbf{S}(\Phi) = \int_0^1 \mathrm{d}t \int_\Lambda \mathrm{d}x\, \bar{\Phi}'(t,x)\big(\partial_t/\nu + \kappa - \Delta/2\big)\Phi'(t,x) + \frac{1}{2}\int_0^1 \mathrm{d}t \int_{\Lambda^2} \mathrm{d}x\,\mathrm{d}y\, |\Phi'(t,x)|^2\, w(x-y)\, |\Phi'(t,y)|^2\,.$$

- Formally deduce $\mathcal{Z}_\nu \to \zeta$ by stationary phase.

Hubbard-Stratonovich transformation

- Let $\mathcal{C} > 0$ be an $n \times n$ matrix. The Gaussian probability measure on $\mathbb{R}^n$ with covariance $\mathcal{C}$ is

$$\mu_{\mathcal{C}}(\mathrm{d}u) := \frac{1}{\sqrt{(2\pi)^n \det \mathcal{C}}} \, \mathrm{e}^{-\frac{1}{2}\langle u, \mathcal{C}^{-1} u\rangle} \, \mathrm{d}u \, .$$

- **Wick's theorem:** for any $f \in \mathbb{R}^n$ we have

$$\int \mu_{\mathcal{C}}(\mathrm{d}u) \, \mathrm{e}^{\mathrm{i}\langle f, u\rangle} = \mathrm{e}^{-\frac{1}{2}\langle f, \mathcal{C} f\rangle} \, .$$

- **Hubbard-Stratonovich transformation:** for a real Gaussian measure $\mu_{\mathcal{C}}$ (not necessarily finite dimensional) with covariance $\mathcal{C}$ we have

$$\int \mu_{\mathcal{C}}(\mathrm{d}\sigma) \, \mathrm{e}^{\mathrm{i}\langle f, \sigma\rangle} = e^{-\frac{1}{2}\langle f, \mathcal{C} f\rangle} \, .$$

(In general formal if f and σ are both rough!)

Functional integral + HS transformation

- Consider $\sigma : [0,\nu] \times \Lambda \to \mathbb{R}$ centred with law $\mu_{\mathcal{C}}$ and covariance

$$\int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\,\sigma(\tau,x)\,\sigma(\tilde{\tau},\tilde{x}) = \nu\,\delta(\tau-\tilde{\tau})\,w(x-\tilde{x}) \equiv \mathcal{C}^{\tau,\tilde{\tau}}_{x,\tilde{x}}\,.$$

- (Formally) use HS with $f = |\Phi|^2$ and let $K(u) := \partial_\tau - \Delta/2 - u$

$$\mathcal{Z}'_\nu := \frac{\int \mathrm{D}\Phi\,\mathrm{e}^{-\mathbf{S}(\Phi)}}{\int \mathrm{D}\Phi\,\mathrm{e}^{-\mathbf{S^0}(\Phi)}} = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\,\frac{\int \mathrm{D}\Phi\,\exp\big(-\langle \Phi, \overbrace{(\partial_\tau - \Delta/2 + \kappa - \mathrm{i}\sigma)}^{K(-\kappa+\mathrm{i}\sigma)}\Phi\rangle\big)}{\int \mathrm{D}\Phi\,\exp\big(-\langle \Phi, (\partial_\tau - \Delta/2 + \kappa)\Phi\rangle\big)}\,.$$

By Gaussian integration

$$\mathcal{Z}'_\nu = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\,\frac{\det K(-\kappa+\mathrm{i}\sigma)^{-1}}{\det K(-\kappa)^{-1}} = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\,\mathrm{e}^{F_1(\sigma)}\,,$$

$$F_1(\sigma) := \int_0^\infty \mathrm{d}t\,\operatorname{Tr}\left(\frac{1}{t+K(-\kappa+\mathrm{i}\sigma)} - \frac{1}{t+K(-\kappa)}\right).$$

We used $\det(A) = \exp\big(\operatorname{Tr}\log A\big), \log a - \log b = -\int_0^\infty \mathrm{d}t\big(\frac{1}{t+a} - \frac{1}{t+b}\big).$

Space-time representation

- **Goal:** Find $K(u)^{-1}$ for $K(u) = \partial_\tau - \Delta/2 - u$.
- We have

$$(K(u)^{-1})^{\tau,\tilde{\tau}}_{x,\tilde{x}} = \sum_{r \in \nu\mathbb{N}} \mathbf{1}_{\tau + r > \tilde{\tau}} \, W^{\tau + r, \tilde{\tau}}_{x,\tilde{x}}(u) \,,$$

where for $[t]_\nu := (t \bmod \nu) \in [0, \nu)$, $(W^{\tau,\tilde{\tau}})_{\tilde{\tau} \leqslant \tau}$ solves

$$\partial_\tau W^{\tau,\tilde{\tau}}(u) = \left(\frac{1}{2}\Delta + u([\tau]_\nu)\right) W^{\tau,\tilde{\tau}}(u) \,, \qquad W^{\tau,\tau}(u) = 1 \,.$$

Feynman-Kac formula: the kernel of $W^{\tau,\tilde{\tau}}$ is given by

$$W^{\tau,\tilde{\tau}}_{x,\tilde{x}}(u) = \int \mathbb{W}^{\tau,\tilde{\tau}}_{x,\tilde{x}}(\mathrm{d}\omega) \, \mathrm{e}^{\int_{\tilde{\tau}}^{\tau} \mathrm{d}t \, u([t]_\nu, \omega(t))} \,.$$

Conclusion: $\mathcal{Z}'_\nu = \int \mathrm{D}\Phi \, \mathrm{e}^{-\mathbf{S}(\Phi)} / \int \mathrm{D}\Phi \, \mathrm{e}^{-\mathbf{S}_0(\Phi)}$ is a rigorous expression in terms of Brownian loops; similarly for correlation functions.
$\rightarrow$ Ginibre representation.

Functional integral representation

- By the Feynman-Kac formula and Hubbard-Stratonovich transformation and working backwards, we formally get

$$\mathcal{Z}_\nu = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \mathrm{e}^{F_1(\sigma)}$$

  (the true quantum partition function).
- In practice, always regularise $\mathcal{C}$.
- Replace $\mathcal{C} \mapsto \mathcal{C}_\eta$ for $\eta > 0$, such that under the law of $\mu_{\mathcal{C}_\eta}$, σ is almost surely smooth.

$$\int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \sigma(\tau, x)\, \sigma(\tilde{\tau}, \tilde{x}) = \nu\, \delta_{\eta,\nu}(\tau - \tilde{\tau})\, w_\eta(x - \tilde{x}) =: (\mathcal{C}_\eta)^{\tau,\tilde{\tau}}_{x,\tilde{x}}\,.$$

- We have

$$\mathcal{Z}_\nu = \lim_{\eta \to 0} \int \mu_{\mathcal{C}_\eta}(\mathrm{d}\sigma)\, \mathrm{e}^{F_1(\sigma)}\,.$$

Wick ordering

- Wick order for $d = 2, 3$.

$$\mathcal{Z}_\nu = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \mathrm{e}^{F_2(\sigma)}$$

where

$$F_2(\sigma) := \int_0^\infty \mathrm{d}t \ \mathrm{Tr}\Bigg(\frac{1}{t + K(-\kappa + \mathrm{i}\sigma)} - \frac{1}{t + K(-\kappa)} \\ - \frac{1}{t + K(-\kappa)}\, \mathrm{i}\sigma\, \frac{1}{t + K(-\kappa)} \Bigg) .$$

- We subtract the first order term in the resolvent expansion.

Functional integral representation: classical setting

- **Classical setting:** derive a similar representation, after Symanzik (1968).
- μ_w: real Gaussian measure with mean zero and covariance

$$\int \mu_w(\mathrm{d}\xi)\, \xi(x)\, \xi(\tilde{x}) = w(x - \tilde{x})\,.$$

- We have

$$\zeta = \int \mu_w(\mathrm{d}\xi)\, \mathrm{e}^{f_2(\xi)}\,,$$

where

$$f_2(\xi) := \int_0^\infty \mathrm{d}t\ \mathrm{Tr}\biggl(\frac{1}{t - \Delta/2 + \kappa - \mathrm{i}\xi} - \frac{1}{t - \Delta/2 + \kappa} - \frac{1}{t - \Delta/2 + \kappa}\, \mathrm{i}\xi\, \frac{1}{t - \Delta/2 + \kappa}\biggr)\,.$$

Conclusion of the proof

- Use the functional integral representations

$$\mathcal{Z}_\nu = \int \mu_{\mathcal{C}}(\mathrm{d}\sigma)\, \mathrm{e}^{F_2(\sigma)}\,, \qquad \zeta = \int \mu_w(\mathrm{d}\xi)\, \mathrm{e}^{f_2(\xi)}$$

  to obtain $\mathcal{Z}_\nu \to \zeta$ as $\nu \to 0$.
- **Fact:** If σ has law $\mu_{\mathcal{C}}$, then

$$\langle \sigma \rangle := \frac{1}{\nu} \int_0^\nu \mathrm{d}\tau\, \sigma(\tau, x)$$

  has law μ_w.
- Show that

$$\lim_{\nu \to 0} \int \mu_{\mathcal{C}}(\mathrm{d}\sigma) \left[\mathrm{e}^{F_2(\sigma)} - \mathrm{e}^{f_2(\langle \sigma \rangle)}\right] = 0\,.$$

- Study Riemann sums and using continuity properties of Brownian paths.
- This analysis can be made **quantitative**: necessary in the proof of the result for Φ_2^4.

Thank you for your attention!