

Representations for Bosonic Ensembles

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Overview

This talk describes some observations I made when trying to relate different approaches to the bosonic ensembles of quantum statistical mechanics.

I learned about cycle statistics and large-deviation principles [Ginibre, Pulé, Dorlas, Ueltschi, Betz, Adams, König, . . .] from Stefan Adams and Wolfgang König.

I also learned a lot from Joel Feldman about the work of T. Bałaban, J. Feldman, H. Knörrer, E. Trubowitz (BFKT) on the interacting Bose gas in the thermodynamic limit.

Original aim: simplify some of the technique of BFKT, find a loop representation in the presence of a condensate.

Related, independent work by Fröhlich, Knowles, Schlein, Sohinger has been reported by Vedran Sohinger at this meeting.

States on operator algebras

$$\begin{split} & \mathtt{H}(\mathtt{a}^{\dagger},\mathtt{a}) = \int_{\mathtt{x}} \mathtt{a}_{\mathtt{x}}^{\dagger}(-\Delta\mathtt{a}_{\mathtt{x}}) + \int_{\mathtt{x},\mathtt{y}} v(\mathtt{x}-\mathtt{y}) : \mathtt{n}_{\mathtt{x}}\mathtt{n}_{\mathtt{y}} : \\ & \text{on Fock space } \mathcal{F} = \bigoplus_{N \geq 0} \mathcal{H}_{N} \\ & Z_{g} = \mathrm{Tr}_{\mathcal{F}} \mathrm{e}^{-\beta(\mathtt{H}-\mu\mathtt{N})} \\ & \omega_{g}(\mathtt{A}) = Z_{g}^{-1} \mathrm{Tr}_{\mathcal{F}} \left[\mathrm{e}^{-\beta(\mathtt{H}-\mu\mathtt{N})} \mathtt{A} \right] \end{split}$$

Lie-Trotter

Coherent-state functional integrals

$$\begin{split} Z_g &= \int \mathrm{e}^{-S(\bar{a}(\tau),a(\tau))} \, \mathcal{D}a \\ S(\bar{a}(\tau),a(\tau)) &= \int_0^\beta \mathrm{d}\tau \; [\bar{a}(\tau)(-\partial_\tau + \mu)a(\tau) + H(\bar{a}(\tau),a(\tau)] \end{split}$$

correlations and connected correlation functions = moments and cumulants of this (formal) measure

\downarrow Projection to \mathcal{H}_N

N-body Schrödinger equation

$$\begin{split} H_N &= \sum_{n=1}^N (-\Delta_n) + \sum_{m < n} v(x_m - x_n) \\ \psi(x_{\pi(1)}, \dots, x_{\pi(N)}) &= \psi(x_1, \dots, x_N) \\ \leftrightarrow \text{Hilbert space} \quad \mathcal{H}_N = \mathbb{S}_N \mathcal{H}_1^{\otimes N} \\ \text{Canonical state} \qquad Z_c = \text{Tr}_{\mathcal{H}_N} e^{-\beta H_N} \end{split}$$

Feynman-Kac

 \downarrow random walk expansion

Interacting Brownian motions

 $Z_c = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} \int \prod_{n=1}^N \mathrm{d} W_{x_n, x_{\pi(n)}}^{(\beta)}(\omega_n) \; \mathrm{e}^{-\sum\limits_{m < n} \int_0^\beta v(\omega_m(\tau) - \omega_n(\tau)) \mathrm{d} \tau}$

sum over permutations $\pi \leftrightarrow$ symmetrisation statistical ensemble of cycles of the permutations π

States on operator algebras

$$\begin{split} & \mathtt{H}(\mathtt{a}^{\dagger},\mathtt{a}) = \int_{\mathtt{x}} \mathtt{a}_{\mathtt{x}}^{\dagger}(-\Delta\mathtt{a}_{\mathtt{x}}) + \int_{\mathtt{x},\mathtt{y}} v(\mathtt{x}-\mathtt{y}) : \mathtt{n}_{\mathtt{x}}\mathtt{n}_{\mathtt{y}} : \\ & \text{on Fock space } \mathcal{F} = \bigoplus_{N \geq 0} \mathcal{H}_{N} \\ & Z_{g} = \mathrm{Tr}_{\mathcal{F}} \mathrm{e}^{-\beta(\mathtt{H}-\mu\mathtt{N})} \\ & \omega_{g}(\mathtt{A}) = Z_{g}^{-1} \mathrm{Tr}_{\mathcal{F}} \left[\mathrm{e}^{-\beta(\mathtt{H}-\mu\mathtt{N})} \mathtt{A} \right] \end{split}$$

Bogoliubov theory

Bose-Einstein Condensation

$$\phi \otimes \ldots \otimes \phi$$

N-body Schrödinger equation

$$\begin{split} H_N &= \sum_{n=1}^N (-\Delta_n) + \sum_{m < n} v(x_m - x_n) \\ \psi(x_{\pi(1)}, \dots, x_{\pi(N)}) &= \psi(x_1, \dots, x_N) \\ \leftrightarrow \text{Hilbert space} \quad \mathcal{H}_N &= \mathbb{S}_N \mathcal{H}_1^{\otimes N} \\ \text{Canonical state} \qquad Z_e = \text{Tr}_{\mathcal{H}_N} e^{-\beta H_N} \end{split}$$

Coherent-state functional integrals

$$\begin{split} Z_g &= \int \mathrm{e}^{-S(\bar{a}(\tau),a(\tau))} \, \mathcal{D}a \\ S(\bar{a}(\tau),a(\tau)) &= \int_0^\beta \mathrm{d}\tau \; [\bar{a}(\tau)(-\partial_\tau + \mu)a(\tau) + H(\bar{a}(\tau),a(\tau)] \end{split}$$

correlations and connected correlation functions = moments and cumulants of this (formal) measure

U(1) symmetry breaking ($a
ightarrow {
m e}^{{
m i} heta} a$)

cycles of length O(N)

Interacting Brownian motions

$$Z_c = \frac{1}{N!} \sum_{\pi \in \mathcal{S}_N} \int \prod_{n=1}^N \mathrm{d} W^{(\beta)}_{x_n, x_{\pi(n)}}(\omega_n) \; \mathrm{e}^{-\sum\limits_{m < n} \int_0^\beta v(\omega_m(\tau) - \omega_n(\tau)) \mathrm{d} \tau}$$

sum over permutations $\pi \leftrightarrow$ symmetrisation statistical ensemble of cycles of the permutations π

Motivations for this work

directly relate and compare:

functional integral and interacting Brownian motion representations criteria for BEC: U(1) breaking and occurrence of infinite cycles

kinetic term $p^2/(2m) \leftrightarrow$ Brownian motion BEC sound wave $c|p| \leftrightarrow$ what kind of process?

Results so far

Functional integral technique: representation of canonical ensemble integrals with real Gaussian measures easy proof of time-continuum limit uniform bounds, no large-field analysis

prove analyticity and decay of correlations by convergent expansions \leftrightarrow persistence of gaps

derive a (stochastic) process in a condensate background

Setup and Notations

Space: any finite (large) graph X, e.g. $X = \eta \mathbb{Z}^d / L\mathbb{Z}^d$. Use notations $\int_x f_x$ for (weighted) sums on X, bilinear form $(f \mid g)_X = \int_x f_x g_x$. $\mathcal{H} = L^2(X, \mathbb{C}), \qquad \mathscr{F}_B^{(N)} = \bigotimes_s^N \mathcal{H}, \qquad \mathscr{F}_B = \bigoplus_{n=0}^\infty \mathscr{F}_B^{(n)}$

The $N\text{-}\mathrm{boson}$ space $\mathscr{F}^{(N)}_B$ has finite dimension $\binom{|\mathbf{X}|+N-1}{|\mathbf{X}|-1}$.

 \mathscr{F}_B is infinite-dimensional even if X is finite.

Vacuum vector $\Omega = (1, 0, 0, \ldots) \in \mathscr{F}_B$.

 \mathbf{P}_N projector to the *N*-particle subspace $\mathscr{F}_B^{(N)}$ of \mathscr{F}_B .

 $\text{CCR algebra } \mathbf{a}_{\mathbf{x}}\mathbf{a}_{\mathbf{y}} - \mathbf{a}_{\mathbf{y}}\mathbf{a}_{\mathbf{x}} = 0, \ \mathbf{a}_{\mathbf{x}}\mathbf{a}_{\mathbf{y}}^{\dagger} - \mathbf{a}_{\mathbf{y}}^{\dagger}\mathbf{a}_{\mathbf{x}} = \eta^{-d}\delta_{\mathbf{x},\mathbf{y}}.$

The local density operators $\mathtt{n}_{\mathtt{x}} = \mathtt{a}_{\mathtt{x}}^{\dagger} \mathtt{a}_{\mathtt{x}} : \mathscr{F}_B^{(\leq N)} \to \mathscr{F}_B^{(\leq N)}$ satisfy

 $\|\mathbf{n}_{\mathbf{x}}\| = N$ and $\mathbf{n}_{\mathbf{x}}\mathbf{n}_{\mathbf{y}} = \mathbf{n}_{\mathbf{y}}\mathbf{n}_{\mathbf{x}}$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.

Hamiltonian

Single particle Hamiltonian: self-adjoint, non-negative operator ${\cal E}$

e.g. the graph Laplacian $-\Delta$, or $-\Delta + W(\mathbf{x})$, W external potential.

Two-body interaction v: a pair of particles at sites x and y contributes $v_{x,y} \in \mathbb{R}$ to the energy. $v_{x,y} = v_{y,x}$. Thus v is a self-adjoint operator on \mathbb{C}^X .

If X is a torus, v is called translation-invariant if $v_{x+z,y+z}=v_{x,y}$ for all z. Then also use the notation $v_{x,y}=v(x-y).$

We will not need translation invariance here.

$$\begin{split} \mathtt{H} &= \mathtt{H}_0 + \mathtt{V} \text{ with } \\ & \mathtt{H}_0 = (\mathtt{a}^\dagger \mid \mathcal{E} \, \mathtt{a})_{\mathrm{X}} \quad \text{ and } \quad \mathtt{V} = \tfrac{1}{2} (\mathtt{n} \mid \mathtt{v} \, \mathtt{n})_{\mathrm{X}} \end{split}$$

The case v = 0 describes free (i.e. noninteracting bosons).

Assume throughout that $v \ge 0$.

 \mathbb{H}_0 and V, hence H, all commute with N, hence map $\mathscr{F}_B^{(N)}$ to itself.

Canonical ensemble

$$\begin{split} \left[\mathbf{F} \right]_{\mathbf{H}}^{(N,\beta,\mathbf{X})} &= \mathrm{Tr}_{\mathscr{F}_B} \left[\mathrm{e}^{-\beta \mathbf{H}} \, \mathbf{F} \, \mathbf{P}_N \right] & (\beta > 0 \text{ the inverse temperature}) \\ Z_c &= Z_c^{(N,\beta,\mathbf{X})} = [\mathbf{1}]^{(N,\beta,\mathbf{X})} & \text{canonical partition function} \\ \left\langle \mathbf{F} \right\rangle_{\mathbf{H}}^{(N,\beta,\mathbf{X})} &= \frac{1}{Z_c} \left[\mathbf{F} \right]_{\mathbf{H}}^{(N,\beta,\mathbf{X})} & \text{canonical expectation value} \end{split}$$

Grand-canonical ensemble

The grand-canonical partition function at chemical potential μ is

$$Z_g^{(\beta,\mu\mathbf{X})} = \sum_{N=0}^{\infty} e^{\beta\mu N} Z_c^{(N,\beta,\mathbf{X})} = \operatorname{Tr}_{\mathscr{F}_B} \left[e^{-\beta(\mathbf{H}-\mu\mathbf{N})} \right] ,$$

and the grand-canonical expectation value is defined similarly.

To explain the integral representation of the Bose ensembles, it is useful to look at a complex field theory first.

Gaussian, a.k.a. Hubbard-Stratonovitch Transform

$$Q + Q^{\dagger} > 0, D = Q^{-1}, \quad \mathrm{d}\mu_D(\bar{\phi}, \phi) = \det Q \,\mathrm{e}^{-(\bar{\phi}, Q\phi)} \,\mathcal{D}\phi, \quad \mathcal{D}\phi = \prod_x \frac{\mathrm{d}\bar{\phi}_x \wedge \mathrm{d}\phi_x}{2\pi \mathrm{i}}.$$

$$Z(\bar{J}, J) = \int d\mu_D(\bar{\phi}, \phi) \, e^{-(|\phi|^2, \, V|\phi|^2)} \, e^{(\bar{J}, \phi) + (\bar{\phi}, J)}$$

Let V > 0, then $e^{-(|\phi|^2, V |\phi|^2)} = \int d\mu_V(h) e^{i(h, |\phi|^2)}$.

$$(h, |\phi|^2) = \int_x \bar{\phi}_x h_x \phi_x = (\bar{\phi}, h\phi)$$

so by Fubini's theorem

$$Z(\bar{J}, J) = \det Q \int d\mu_V(h) \int e^{-(\bar{\phi}, (Q-ih)\phi) + (\bar{J}, \phi) + (\bar{\phi}, J)} \mathcal{D}\phi$$

= $\int d\mu_V(h) e^{(\bar{J}, (Q-ih)^{-1}J)} \det(Q(Q-ih)^{-1}).$

and e.g.

$$\left\langle \bar{\phi}_x \phi_y \right\rangle = \frac{1}{Z} \frac{\delta^2 Z(\bar{J},J)}{\delta J_y \delta \bar{J}_y} \Big|_{\bar{J}=J=0} = \int \mathrm{d}\mu_V(h) \left[(Q-\mathrm{i}h)^{-1} \right]_{x,y} \,\mathrm{e}^{-\mathrm{Tr}\log(1-\mathrm{i}hD)} \,.$$

Formal expansion with loop-vertices

$$-\operatorname{Tr}\log(1-\mathrm{i}hD) = \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr}\left[(\mathrm{i}hD)^{n}\right]$$

so $\mathrm{Tr}\log$ generates 'loop-vertices', and

$$(Q - ih)^{-1} = D + DihD + DihDihD + \dots$$

generates 'path-vertices' for the integral over h.

Uniformity

$$Q = P - iR$$
, $P = P^{\dagger} > 0$ and $R = R^{\dagger}$. So $P = B^2$ with $B = B^{\dagger} > 0$, and

$$Q - ih = B [1 - iC(h + R)C] B$$
. $C = B^{-1}$

Thus $(Q - ih)^{-1}$ exists and is bounded uniformly in h. The Neumann expansion can be done to finite order, with a remainder term that is uniformly bounded in h. Since

$$-\frac{\delta}{\delta h_x} \operatorname{Tr} \log(1 - \mathrm{i}hD) = \left[(1 - \mathrm{i}hD)^{-1} \mathrm{i}D \right]_{x,x} = \mathrm{i} \left[(Q - \mathrm{i}h)^{-1} \right]_{x,x} ,$$

and higher derivatives with respect to h are uniformly bounded in h, the Brydges-Kennedy formula gives a convergent expansion for the correlation functions, provided that $D = Q^{-1}$ is regular.

We will now see that integral representations hold for the quantum many-boson ensembles, with similar uniformity properties.

Double-Gaussian integral representation

Theorem 1 Assume that $v \ge 0$. Set $h_0 = 0$. For $j, j' \in \{0, \dots, \ell\}$ let

$$\mathcal{Q}(h)_{j,j'} = \delta_{j,j'} \ \mathbf{1} - \delta_{j+1,j'} \ \mathrm{e}^{-\epsilon \mathcal{E}} \ \mathrm{e}^{\mathrm{i}\sqrt{\epsilon} \ h_{j'}}$$

Then $Z_c^{(N,\beta,{\rm X})} = \lim_{\ell \to \infty} Z_c^{(N,\beta,{\rm X},\ell)}$ where

$$Z_c^{(N,\beta,\mathbf{X},\ell)} = \lim_{R \to \infty} \int \mathrm{d}\mu_{\mathbb{V}}(h) \int_{\forall x: |a_x| \le R} \mathcal{D}a \, \mathrm{e}^{-(\bar{a}|\mathcal{Q}(h)|a)_{\mathbb{X}}} \frac{(\bar{a}_{\ell}|a_0)_{\mathbf{X}}^N}{N!}$$

If $\mathcal{E} > 0$, then $\operatorname{Re}(\overline{a} \mid \mathcal{Q}(h) \mid a)_{\mathbb{X}}$ is strictly positive, so the integral over a converges absolutely and $R \to \infty$ can be taken.

$$Z_c^{(N,\beta,\mathcal{X},\ell)} = \int \mathrm{d}\mu_{\mathbb{V}}(h) \int_{\mathbb{C}^{\mathbb{X}}} \mathcal{D}a \, \mathrm{e}^{-(\bar{a}|\mathcal{Q}(h)|a)_{\mathbb{X}}} \, \frac{(\bar{a}_{\ell}|a_0)_{\mathcal{X}}}{N!}$$

$${}^{1}\mathcal{D}a = \prod_{j=0}^{\ell} \mathrm{d}^{\mathrm{X}}a_{j} = \prod_{j=0}^{\ell} \prod_{\mathrm{x}\in\mathrm{X}} \frac{\mathrm{d}\bar{a}_{j,\mathrm{x}} \wedge a_{j,\mathrm{x}}}{2\pi\mathrm{i}} \qquad \mathrm{d}\mu_{\mathbb{V}}(h) = \prod_{j=1}^{\ell} \mathrm{d}\mu_{\mathrm{v}}(h_{j})$$

Covariance

The operator $\mathcal{Q}(h)$ is upper triangular: setting $\mathtt{A}_{\jmath} = \mathrm{e}^{-\epsilon \mathcal{E}} \, \mathrm{e}^{\mathrm{i} \sqrt{\epsilon} h_{\jmath}}$

$$\mathcal{Q}(h) = \begin{bmatrix} 1 & -A_1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -A_2 & 0 & \dots & 0 \\ & & \ddots & & \ddots & \\ 0 & 0 & 0 & 1 & -A_\ell \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \qquad \epsilon = \frac{\beta}{\ell}$$

The covariance $\mathcal{C}(h)=\mathcal{Q}(h)^{-1}$ exists, is upper triangular, and given by a terminating Neumann series.

C(h) is entire analytic in h. In fact, for every finite ℓ , it is a trigonometric polynomial in h.

Note that $\det C(h) = 1$ and that Q has no periodic boundary condition in time.

Remarks on the proof

Use the Lie product formula $e^{-\beta H} \underset{\ell \to \infty}{\longleftarrow} \left(e^{-\frac{\beta}{\ell} H_0} e^{-\frac{\beta}{\ell} V} \right)^{\ell}$. Recall the notation $\epsilon = \frac{\beta}{\ell}$. All n_x commute, so by the spectral theorem

$$e^{-\epsilon \mathbf{V}} = e^{-\epsilon(\mathbf{n}|\mathbf{v}\mathbf{n})_{\mathbf{X}}} = \int d\mu_{\mathbf{v}}(h) \ e^{i\sqrt{\epsilon}(h|\mathbf{n})_{\mathbf{X}}} = \int d\mu_{\mathbf{v}}(h) \ e^{(\mathbf{a}^{\dagger}|\ i\sqrt{\epsilon}h\ \mathbf{a})_{\mathbf{X}}}$$

 $d\mu_v = normalized$, centered Gaussian measure on \mathbb{R}^X with covariance v. Coherent states $\gamma(a) = e^{(a|\mathbf{a}^{\dagger})_X} \Omega$ with the properties²

$$\begin{split} \mathbf{a}_{\mathbf{x}}\gamma(a) &= a_{\mathbf{x}}\gamma(a) \qquad \mathrm{e}^{(\mathbf{a}^{\dagger}|\mathcal{K}|\mathbf{a})_{\mathbf{X}}}\gamma(a) = \gamma(\mathrm{e}^{\mathcal{K}}a) \quad \text{if } \mathrm{e}^{\mathcal{K}} \text{ is bounded} \\ 1_{\mathscr{F}_{B}} &= \lim_{R \to \infty} \int_{\mathbb{C}_{R}^{\mathbf{X}}} \mathrm{d}^{\mathbf{X}}a \ |\kappa_{a}\rangle\langle\kappa_{a}| \qquad \mathrm{Tr}_{\mathscr{F}_{B}}\mathbf{A} = \lim_{R \to \infty} \int_{\mathbb{C}_{R}^{\mathbf{X}}} \mathrm{d}^{\mathbf{X}}a \ \langle\kappa_{a} \mid \mathbf{A} \ \kappa_{a}\rangle \\ &\forall a, a' \in \mathbb{C}^{\mathbf{X}} \qquad \langle\gamma(a') \mid \mathbf{P}_{N}\gamma(a)\rangle = \frac{1}{N!} \left(\bar{a}' \mid a\right)_{\mathbf{X}}^{N}. \end{split}$$

$${}^{2}\mathbb{C}_{R} = \{ z \in \mathbb{C} : |z| \leq R \}$$

The oscillatory a-integral

Integration over h leaves over an integral over a,

$$Z_c = \int \mathcal{D}a \, \mathrm{e}^{-(\bar{a}|\mathcal{Q}(h) \, a)_{\mathbb{X}} - \mathcal{V}(a)} \, \frac{(\bar{a}_\ell | a_0)_{\mathrm{X}}^N}{N!}$$

Both the kinetic and the interaction term are complex-valued – this is a complex oscillatory integral.

For $v_{x,y} = v \delta_{x,y}$, the interaction is $\mathcal{V}(a) = \epsilon \sum_{j} \int_{x} \Psi(\bar{a}_{j-1,x} a_{j,x})$, where

$$\Psi(z) = -\ln \Phi(z), \qquad \Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} e^{-\frac{1}{2} \epsilon v n^2}$$

Thus the positivity of the operator V is lost here.

We will see how to recover it shortly.

The h-representation

The *a*-integral in $Z_c = \int d\mu_{\mathbb{V}}(h) \int \mathcal{D}a \ e^{-(\bar{a}|\mathcal{Q}(h)|a)_{\mathbb{X}}} \frac{(\bar{a}_{\ell}|a_0)_{\mathbb{X}}^N}{N!}$ is Gaussian as well. At $\epsilon > 0$ it is absolutely convergent. The factor $(\bar{a}_{\ell} \mid a_0)_{\mathbb{X}}^N$ leads to a permanent:

$$Z_c^{(N,\beta,\mathbf{X},\ell)} = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_{N_{\mathbf{X}_1,\ldots,\mathbf{X}_N}}} \int \mathrm{d}\mu_{\mathbb{V}}(h) \prod_{k=1}^N \mathcal{C}(h)_{(0,\mathbf{X}_k),(\ell,\mathbf{X}_{\sigma(k)})}$$

Denote $\langle F(h)\rangle_h=\int F(h)\mathrm{d}\mu_{\mathbb{V}}(h)$ and

$$\mathscr{P}_{N,\mathcal{X}}\mathcal{C}_{j,j'} = \sum_{\pi \in \mathscr{S}_N} \int_{\mathbf{x}_1, \dots, \mathbf{x}_N} \prod_{n=1}^N \mathcal{C}_{(j,\mathbf{x}_n), (j', \mathbf{x}_{\pi(n)})} \ .$$

Then

$$Z_{c}^{(N,\beta,\mathbf{X},\ell)} = \frac{1}{N!} \left\langle \mathscr{P}_{N,\mathbf{X}} \mathcal{C}(h)_{0,\ell} \right\rangle_{h}$$

The random walk representation

Recall
$$\mathcal{Q}(h)_{j,j'} = \delta_{j,j'} \ \mathbf{1} - \delta_{j+1,j'} \ \mathrm{e}^{-\epsilon \mathcal{E}} \ \mathrm{e}^{\mathrm{i}\sqrt{\epsilon} \ h_{j'}}.$$

The Neumann series for $\mathcal{C}(h)=\mathcal{Q}(h)^{-1}$ corresponds to a random walk expansion

$$\mathcal{C}(h)_{(0,\mathbf{x}_k),(\ell,\mathbf{x}_{\sigma(k)})} = \sum_{\mathbf{y}^{(k)}} \prod_{j=1}^{\ell} \left(e^{-\epsilon \mathcal{E}} \right)_{\mathbf{y}_{j-1}^{(k)},\mathbf{y}_j^{(k)}} e^{-i\sqrt{\epsilon}h_j(\mathbf{y}_j^{(k)})}$$

where $y^{(k)} = (y^{(k)}_0, \dots y^{(k)}_\ell)$ is a walk $x_k \to x_{\sigma(k)}$ with transition amplitude

$$P(\mathbf{y}^{(k)}) = \prod_{j=1}^{\ell} \left(e^{-\epsilon \mathcal{E}} \right)_{\mathbf{y}_{j-1}^{(k)}, \mathbf{y}_{j}^{(k)}}$$

An easy exercise in Gaussian integration gives

Theorem 2

$$Z_{c}^{(N,\beta,\mathbf{X},\ell)} = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_{N}} \int_{\mathbf{x}_{1},..,\mathbf{x}_{N}} \sum_{\substack{\mathbf{y}^{(1)},..,\mathbf{y}^{(N)}\\\mathbf{y}^{(k)}:\mathbf{x}_{k} \to \mathbf{x}_{\sigma}(k)}} e^{-\mathcal{V}(\mathbf{y}^{(1)},..,\mathbf{y}^{(N)})} \prod_{k=1}^{N} P(\mathbf{y}^{(k)})$$

with³

$$\mathcal{V}(\mathbf{y}^{(1)},..,\mathbf{y}^{(N)}) = \frac{1}{2} \sum_{1 \le k,k' \le N} \int_{\tau} \mathbf{v} \left(\mathbf{y}^{(k)}(\tau) - \mathbf{y}^{(k')}(\tau) \right) \ .$$

This is the random walk representation that replaces the interacting Brownian motion for $\epsilon > 0$. It converges to the IBM representation as $\epsilon \to 0$.

$${}^{3}\int_{\tau} F(\tau) = \epsilon \sum_{j} F(\epsilon j)$$

Uniformity

Lemma 1 If \mathcal{E} generates a stochastic process (i.e. $(e^{-\tau \mathcal{E}})_{xy} \ge 0$ for all $\tau > 0$ and all x, y), then for all $j' \ge j$, all $x, x' \in X$, and all h

$$\left|\mathcal{C}(h)_{(j,\mathbf{x}),(j',\mathbf{x}')}\right| \le \left|\mathcal{C}(0)_{(j,\mathbf{x}),(j',\mathbf{x}')}\right| = \left(e^{(j'-j)\epsilon \mathcal{E}}\right)_{\mathbf{x},\mathbf{x}'}$$

•

Proof. Writing out the matrix products

$$\mathcal{C}(h)_{(j,\mathbf{x}),(j',\mathbf{x}')} = \sum_{\mathbf{y}:\mathbf{x}\to\mathbf{x}'} \prod_{j=j+1}^{j'} \left(e^{-\epsilon\mathcal{E}} \right)_{\mathbf{y}_{j-1},\mathbf{y}_j} e^{i\sqrt{\epsilon}h_{j,\mathbf{y}_j}}$$

By hypothesis

$$\left|\mathcal{C}(h)_{(j,\mathbf{x}),(j',\mathbf{x}')}\right| \leq \sum_{\mathbf{y}:\mathbf{x}\to\mathbf{x}'} \prod_{j=j+1}^{j'} \left(e^{-\epsilon\mathcal{E}}\right)_{\mathbf{y}_{j-1},\mathbf{y}_{j}} = \mathcal{C}(0)_{(j,\mathbf{x}),(j',\mathbf{x}')}$$

An absolutely convergent a-integral

Theorem 3 Let v be translation invariant, $\mathcal{E} > 0$ and $v \ge 0$, and \mathcal{E} generate a stochastic process. Then, up to an explicit constant, $Z_c^{(N,\beta,X)}$ is the limit $\ell \to \infty$ of

$$\mathfrak{Z}_{\ell}^{(N,\beta,\mathrm{X})} = \int \mathcal{D}a \, \mathrm{e}^{-\mathcal{S}_{\mathrm{X}}(a)} \, \frac{1}{N!} (\bar{a}(\beta) \mid a(0))_{\mathrm{X}}^{N}$$

with the action $\mathcal{S}_{\mathbb{X}}$ given by

$$\mathcal{S}_{\mathbb{X}}(a) = \int_{\tau} \left\{ (\bar{a}(\tau) \mid \left[(-\partial_{\tau} + \mathbf{v}(0) + \mathcal{E}^{(\epsilon)})a \right](\tau))_{\mathbf{X}} + \left(|a(\tau)|^2 \right] \mathbf{v} \mid a(\tau)|^2 \right)_{\mathbf{X}} \right\}$$

Here ∂_{τ} denotes the discrete forward time derivative

$$(\partial_{\tau}a)(\tau,\mathbf{x}) = \frac{1}{\epsilon}(a(\tau+\epsilon,\mathbf{x}) - a(\tau,\mathbf{x}))$$

with the boundary condition $a(\beta + \epsilon, \mathbf{x}) = 0$, and

$$(\mathcal{E}^{(\epsilon)}a)(\tau,\mathbf{x}) = \frac{1}{\epsilon} \int_{\mathbf{y}} (1 - e^{-\epsilon\mathcal{E}})(\mathbf{x} - \mathbf{y}) \ a(\tau + \epsilon, \mathbf{y}) \ .$$

Remarks

If v > 0, the quartic term is strictly positive, so the integral converges absolutely. The limit remains unchanged if $(\mathcal{E}^{(\epsilon)}a)(\tau, x)$ is replaced by $(\mathcal{E}a)(\tau, x)$ in the action. The v(0) can be removed by normal ordering of V.

In the translation-invariant case

$$S_{\mathbb{X}}(a) = \int_{\tau} \int_{\mathbf{x}} \bar{a}(\tau, \mathbf{x}) \left[(-\partial_{\tau} + \mathbf{v}(0) + \mathcal{E}^{(\epsilon)}) a \right](\tau, \mathbf{x}) \\ + \int_{\tau} \int_{\mathbf{x}, \mathbf{y}} |a(\tau, \mathbf{x})|^2 |\mathbf{v}(\mathbf{x} - \mathbf{y})| |a(\tau, \mathbf{y})|^2$$

Idea of the proof

The main idea is to avoid estimating oscillatory integrals and instead use the h-representation.

$$\mathrm{e}^{-\epsilon(|a_{j}|^{2}|\mathbf{v}||a_{j}|^{2})} = \int \mathrm{d}\mu_{\mathbf{v}}(h_{j}) \, \mathrm{e}^{\mathrm{i}\sqrt{\epsilon}(h_{j}||a_{j}|^{2})}$$

gives a quadratic form

$$\mathcal{Q}_2(h)_{j,j'} = \delta_{j,j'} \left(1 - \mathrm{i}\sqrt{\epsilon} h_j \right) - \delta_{j+1,j'} \mathrm{e}^{-\epsilon\tilde{\mathcal{E}}}$$

Taking out the factor from the diagonal gives

$$\mathcal{Q}_1(h)_{j,j'} = \delta_{j,j'} - \delta_{j+1,j'} e^{-\epsilon \tilde{\mathcal{E}}} (1 - i\sqrt{\epsilon} h_j)^{-1}$$

Observe that

$$(1 - i\sqrt{\epsilon} h_j)^{-1} - e^{i\sqrt{\epsilon}h_j} = O(\epsilon)$$

By the resolvent identity, the uniform bounds for C and C_1 , and an additional integration by parts, one can estimate the difference of permanents of C and C_1 .

Grand-canonical ensemble

Recall that
$$Z_g^{(eta,\mu\mathrm{X})} = \sum\limits_{N=0}^\infty \mathrm{e}^{eta\mu N} Z_c^{(N,eta,\mathrm{X})}.$$

Absorb the prefactor by a shift $\mathcal{E} \to \mathcal{E}_{\mu} = \mathcal{E} - \mu$. Then resum

$$\sum_{N=0}^{\infty} \frac{1}{N!} (\bar{a}_{\ell} \mid a_0)_{\mathrm{X}}^N = \mathrm{e}^{(\bar{a}_{\ell} \mid a_0)_{\mathrm{X}}}$$

This produces the standard periodic boundary condition in the quadratic form of the a fields, hence the standard time-ordered Green function, and

$$Z_g^{(N,\beta,\mathcal{X},\ell)} = \int \mathrm{d}\mu_{\mathbb{V}}(h) \int_{\mathbb{C}^{\mathbb{X}}} \mathcal{D}a \, \mathrm{e}^{-(\bar{a}|\mathcal{K}(h)|a)_{\mathbb{X}}}$$

with

$$\mathcal{K}(h)_{j,j'} = \mathcal{Q}(h)_{j,j'} - \delta_{j,\ell} \delta_{0,j'} \mathbf{1} .$$

Theorem 4 Assume that $\mathcal{E}_{\mu} > 0$. Then

the integral for the grand canonical partition function $Z_g^{(N,\beta,X,\ell)}$ is absolutely convergent. The grand-canonical covariance $\mathcal{G}(h) = \mathcal{K}(h)^{-1}$

- exists for all h, and has a norm bounded uniformly in h
- is analytic in h if $|\text{Im } h_{j,x}| < \sqrt{\epsilon}e_{min}$, where e_{min} is the smallest eigenvalue of \mathcal{E}_{μ} .
- If \mathcal{E}_{μ} generates a stochastic process, then for all $j' \geq j$, all $x, x' \in X$, and all h,

$$\forall h \in \mathbb{R}^{\mathbb{X}} : \quad \left| \mathcal{G}(h)_{(j,\mathbf{x}),(j',\mathbf{x}')} \right| \leq \mathcal{G}(0)_{(j,\mathbf{x}),(j',\mathbf{x}')} .$$

Comparing canonical and grand-canonical

the canonical covariance C(h) has only forward propagation in time the grand-canonical G(h) has forward- and backward propagation

 $\det \mathcal{C}(h) = 1$ but $\det \mathcal{G}(h) \neq 1$.

In Feynman graph expansions this implies the absence of loop vertices in the canonical ensemble (fixed particle number implies absence of pair creation!)



Generating functional for correlations

Let $m = \mu + i\nu$ be a complex chemical potential and

$$\mathcal{K}_{\mathrm{i}\nu}(\mathcal{E}_{\mu},h) = 1 - \mathcal{N} e^{-\epsilon \mathcal{E}} e^{\mathrm{i}\sqrt{\epsilon}h} - \mathcal{R} e^{\mathrm{i}\beta\nu} , \qquad \mathcal{G}_{\mathrm{i}\nu}(\mathcal{E}_{\mu},h) = \mathcal{K}_{\mathrm{i}\nu}(\mathcal{E}_{\mu},h)^{-1}$$

Partition function with sources J, \bar{J}

$$Z_{g}^{(m,\beta,\mathbf{X},\ell)}(\bar{J},J) = \left\langle \int e^{-(\bar{a}|\mathcal{K}_{i\nu}(\mathcal{E}_{\mu},h)a)_{\mathbf{X}} + (\bar{J}|a)_{\mathbf{X}} + (J|\bar{a})_{\mathbf{X}}} \mathcal{D}a \right\rangle_{h}$$
$$= \left\langle \int e^{(\bar{J}|\mathcal{G}_{i\nu}(\mathcal{E}_{\mu},h)J)_{\mathbf{X}}} \det \mathcal{G}_{i\nu}(\mathcal{E}_{\mu},h) \right\rangle_{h}$$
$$= \left\langle \int e^{\mathcal{W}_{0}(\bar{J},J,h)} \right\rangle_{h}$$

with

$$\mathcal{W}_{0}(\bar{J}, J, h) = (\bar{J} \mid \mathcal{G}_{i\nu}(\mathcal{E}_{\mu}, h)J)_{X} + \operatorname{Tr} \ln \mathcal{C} - \operatorname{Tr} \ln(1 - e^{i\beta\nu}\mathcal{R}\mathcal{C})$$
$$\mathcal{C} = (1 - \mathcal{N}e^{-\epsilon\mathcal{E}}e^{i\sqrt{\epsilon}h})^{-1}$$

Decay of Correlations

Theorem 5 Assume that space X is a regular lattice, that \mathcal{E} generates a stochastic process and $\mathcal{E} \ge 0$, and that the covariance for v = 0 (noninteracting particles) has exponential decay. Assume that $v_{x,y}$ also decays exponentially in |x - y|.

There is $\mu_0 < 0$ s.t. for all $\mu = \text{Re } m \le \mu_0$ and all sufficiently small |v| $Z_g^{(m,\beta,\mathbf{X},\ell)}(\bar{J},J)$ is analytic in m,β,\bar{J},J

the connected correlation functions

$$G_{2m}(x_1, \dots, x_n; x'_1, \dots, x'_n) = \frac{\delta^{2n}}{\delta J_{\ell, x_1} \dots \delta \bar{J}_{0, x'_n}} \ln Z_g(\bar{J}, J) \mid_{J=\bar{J}=0}$$

have exponential tree graph decay, uniform in X and in ℓ .

Cycle Summations

Every permutation contributing in the sum for the permanent consists of $K \ge 1$ cycles with lengths $\kappa_1, \ldots, \kappa_K$. Then

$$\frac{1}{N!} \mathscr{P}_{N,\mathrm{X}} \left[\mathcal{C}(\mathcal{E}, \mathrm{h})_{0,\ell} \right] = \sum_{K=1}^{N} \frac{1}{K!} \sum_{\substack{\kappa_1, \dots, \kappa_K \in \mathbb{N} \\ \kappa_1 + \dots + \kappa_K = N}} \prod_{k=1}^{K} \frac{1}{\kappa_k} \operatorname{Tr} \left(\mathcal{C}(\mathcal{E}, \mathrm{h})_{0,\ell}^{\kappa_k} \right)$$

With this, the grand canonical partition function becomes

$$\begin{split} Z_{g}^{(\beta,\mu+\mathrm{i}\nu,\mathrm{X},\ell)}(\mathrm{H}) &= \left\langle \sum_{K=1}^{\infty} \frac{1}{K!} \sum_{\kappa_{1},\ldots,\kappa_{K}\in\mathbb{N}} \prod_{k=1}^{K} \frac{\mathrm{e}^{\mathrm{i}\beta\nu\kappa_{k}}}{\kappa_{k}} \operatorname{Tr}\left(\mathcal{C}(\mathcal{E}_{\mu},\mathrm{h})_{0,\ell}^{\kappa_{k}}\right)\right\rangle_{h} \\ &= \left\langle \sum_{K=1}^{\infty} \frac{1}{K!} \left(-\operatorname{Tr}\ln(1-\mathrm{e}^{\mathrm{i}\beta\nu}\mathcal{C}(\mathcal{E}_{\mu},\mathrm{h})_{0,\ell})\right)^{K}\right\rangle_{h} \\ &= \left\langle \operatorname{e}^{-\operatorname{Tr}\ln(1-\mathrm{e}^{\mathrm{i}\beta\nu}\mathcal{C}(\mathcal{E}_{\mu},\mathrm{h})_{0,\ell})}\right\rangle_{h} \\ &= \left\langle \det(1-\mathrm{e}^{\mathrm{i}\beta\nu}\mathcal{C}(\mathcal{E}_{\mu},\mathrm{h})_{0,\ell})^{-1}\right\rangle_{h} \end{split}$$

Cycle Summations

Thus the cycle expansion is simply an expansion of $\operatorname{Tr}\log$, as it arises from integrating out the *a* fields,

$$Z_g^{(\beta,\mu+\mathrm{i}\nu,\mathrm{X},\ell)}(\mathrm{H}) = \langle \det \mathcal{G}(\mathcal{E},\mathrm{h}) \rangle_h$$

namely, the expansion in the term that creates the periodic boundary condition.

As long as this expansion converges, the expected cycle length is finite.

Thus: infinite cycles occur when one reaches the convergence radius of the expansion.

Does this also correspond to a phase transition?

It does, if the closest singularity is on the positive real axis.

Condensate and Bogoliubov modes

Orthogonal decomposition a = b + c

 $c = (c_{\tau})_{\tau \in \mathbb{T}}$ is independent of x and $(1 \mid b(\tau))_{\mathrm{X}} = \int_{\mathrm{x}} b(\tau, \mathrm{x}) = 0$.

The canonical partition function then becomes

$$\mathfrak{Z}_{\ell} = |\mathbf{X}|^{N} \sum_{K=0}^{N} \frac{1}{(N-K)!} \int \mathcal{D}^{\mathbb{T}} c \, \mathrm{e}^{-\mathcal{A}_{0}(c)} \left(\overline{c(\beta)} \, c(0)\right)^{N-K} \, \mathcal{Y}_{\ell}^{(\beta,K,\mathbf{X})}(c)$$

with

$$\mathcal{Y}_{\ell}^{(\beta,K,\mathbf{X})}(c) = \frac{1}{K!} \int \mathcal{D}' b \, \mathrm{e}^{-\mathcal{A}_2(b,c) - \mathcal{A}_3(b,c) - \mathcal{A}_4(b)} \left(\frac{1}{|\mathbf{X}|} \int_{\mathbf{x}} \overline{b(\beta,\mathbf{x})} b(0,\mathbf{x}) \right)^K$$

Summation over $K \leftrightarrow$ the condensate plays the role of a particle reservoir: if $K \ll N$, the *b*-subsystem becomes effectively grand canonical.

Quadratic form and Bogoliubov spectrum

The action

$$\mathcal{A}_{2}(b,c) = \frac{\mathbf{v}}{2} \int_{\tau,\mathbf{x}} \left[4|c(\tau)|^{2} |b(\tau,\mathbf{x})|^{2} - c(\tau)^{2} \overline{b(\tau,\mathbf{x})}^{2} - \overline{c(\tau)}^{2} b(\tau,\mathbf{x})^{2} \right] \\ + \int_{\tau} (\bar{b}(\tau) \mid (-\partial_{\tau} + \mathcal{E}) b(\tau))_{\mathbf{X}}$$

is a c-dependent quadratic form in b.

The lower eigenvalue exhibits the Bogoliubov spectrum: for time-independent $c = c_0$ and kinetic term p^2 in the continuum limit,

$$E_B(\mathbf{p}) = |\mathbf{p}|\sqrt{\mathbf{w}^2 + \mathbf{p}^2}$$

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where $w^2 = 2v|c_0|^2$.

Positivity and slow decay

Lemma 2 For all $t \ge 0$, the x-space kernel of $e^{-t|p|\sqrt{w^2+p^2}}$ is positive.

The decay is, however, slow since already

$$e^{-t|\mathbf{p}|}(\mathbf{x},\mathbf{y}) = \begin{cases} \frac{1}{\pi^2} \frac{t}{(t^2+|\mathbf{x}-\mathbf{y}|^2)^2} & \text{for } d=3\\ \Gamma(\frac{d+1}{2}) \pi^{-\frac{d+1}{2}} \frac{t}{(t^2+|\mathbf{x}-\mathbf{y}|^2)^{\frac{d+1}{2}}} & \text{for } d>3 \end{cases}.$$
 (1)

This slow decay suggests that a stochastic representation as a random walk in a fluctuating condensate background has long jumps, hence differs from the one corresponding to Brownian motion.

The cubic term A_3 also leads to branching and coalescence, corresponding to processes where one of two scattering particles emerges from, or gets absorbed in, the condensate.

Concluding comments

The coherent-state functional integral has a rigorous mathematical justification.

It allows to derive the interacting random walk (Brownian motion) representation straightforwardly.

The proof of the existence of the time-continuum limit, with a variety of actions, for many-boson functional integrals is now as simple as in the fermionic case.

The method, in particular the uniform bounds for the covariance, also allow to prove decay of correlations.

Standard random-walk expansions converge only in the unbroken phase.

A similar expansion in presence of a condensate involves processes with branching and coalescence, and long-range jumps. This creates infrared divergences in expansions, which make renormalization necessary.