Dimerisation in quantum spin chains

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Collaboration with **Jakob Björnberg, Peter Mühlbacher, Bruno Nachtergaele**

Quantum spin chain

Chain $\{-\ell+1, ..., \ell\}$, with 2ℓ sites

Hilbert space $\mathcal{H}_\ell = (\mathbb{C}^n)^{\otimes 2\ell}$, $n = 2S+1$

Hamiltonian $H_\ell = \displaystyle\sum_{x=-\ell+1}^{\ell-1} \big(uT_{x,x+1} + vQ_{x,x+1}\big)$ where

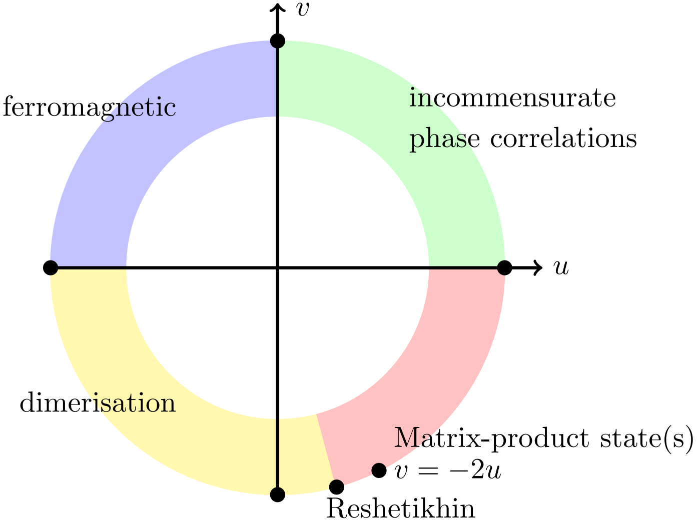
- T is the transposition operator, $T(\phi \otimes \varphi) = \varphi \otimes \phi$
- Q is the projection onto the vector

$$\frac{1}{\sqrt{n}} \sum_{\alpha=1}^{n} e_\alpha \otimes e_\alpha,$$

This is the general form of two-body interactions with $O(n)$ symmetry

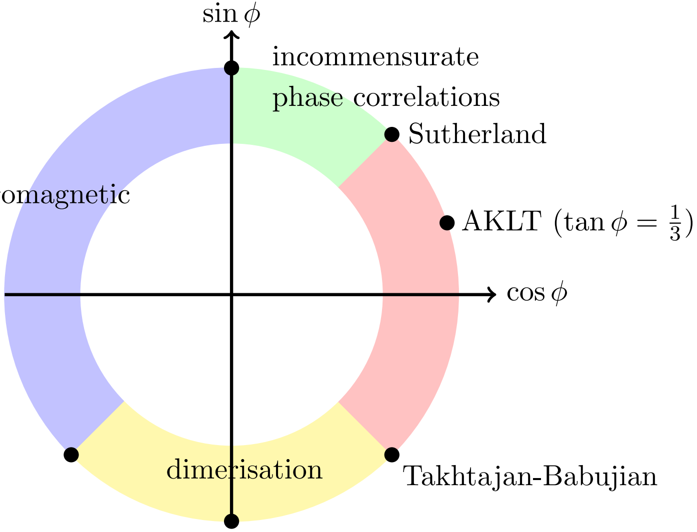
Expected phase diagram (for $n \geqslant 3 \iff S \geqslant 1$)

Interactions $uT_{x,x+1} + vQ_{x,x+1}$



Case $n = 3 \Longleftrightarrow S = 1$

Interact. $\cos\phi\, \vec{S}_x \cdot \vec{S}_{x+1} + \sin\phi\, (\vec{S}_x \cdot \vec{S}_{x+1})^2 = 3(\sin\phi - \cos\phi) P + \cos\phi\, T$



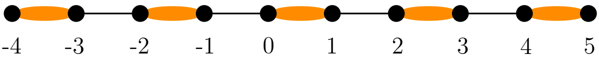
Dimerisation

Let $L^{\alpha,\alpha'} = |\alpha\rangle\langle\alpha'| - |\alpha'\rangle\langle\alpha|$ be the generators of Lie algebra $\mathfrak{o}(n)$

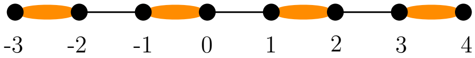
Theorem 1 **[Björnberg, Mühlbacher, Nachtergaele, U '21]**
$\exists\, n_0, u_0, c > 0$ (independent of ℓ) such that for $n > n_0$ and $|u| < u_0$, we have that for all $1 \leqslant \alpha < \alpha' \leqslant n$,

$$\lim_{\beta\to\infty} \Big[\langle L_0^{\alpha,\alpha'} L_1^{\alpha,\alpha'} \rangle_{\ell,\beta,u} - \langle L_{-1}^{\alpha,\alpha'} L_0^{\alpha,\alpha'} \rangle_{\ell,\beta,u} \Big] > c \qquad \text{for all } \ell \text{ odd}$$

$$\lim_{\beta\to\infty} \Big[\langle L_0^{\alpha,\alpha'} L_1^{\alpha,\alpha'} \rangle_{\ell,\beta,u} - \langle L_{-1}^{\alpha,\alpha'} L_0^{\alpha,\alpha'} \rangle_{\ell,\beta,u} \Big] < -c \qquad \text{for all } \ell \text{ even}$$



$\ell = 5$, odd



$\ell = 4$, even

Remark: **[Aizenman, Duminil-Copin, Warzel '20]** proved dimerisation at $(u,v) = (0,-1)$ for all $S \geqslant 1$

Exponential decay of spin-spin correlations

Theorem 2 **[Björnberg, Mühlbacher, Nachtergaele, U '21]**
There exist $n_0, u_0, c_1, c_2, C > 0$ (independent of ℓ) such that for $n > n_0$ and $|u| < u_0$, we have

$$\lim_{\beta \to \infty} \left| \langle L_x^{\alpha,\alpha'} \, \mathrm{e}^{-tH_\ell} \, L_y^{\alpha,\alpha'} \, \mathrm{e}^{tH_\ell} \rangle_{\ell,\beta,u} \right| \leqslant C \, \mathrm{e}^{-c_1|x-y| - c_2|t|}$$

for all $\ell \in \mathbb{N}$, all $x, y \in \{-\ell+1, \dots, \ell\}$, all $1 \leqslant \alpha < \alpha' \leqslant n$, and all $t \in \mathbb{R}$

Gap in energy spectrum

Let $E_0^{(\ell)} < E_1^{(\ell)} < \dots$ be the eigenvalues of H_ℓ. The gap is defined as

$$\Delta^{(\ell)} = E_1^{(\ell)} - E_0^{(\ell)}$$

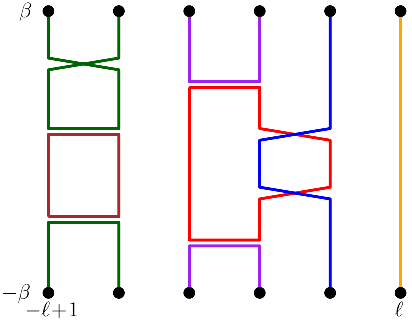
Theorem 3 **[Björnberg, Mühlbacher, Nachtergaele, U '21]**
There exist constants $n_0, u_0, c > 0$ (independent of ℓ) such that for $n > n_0$ and $|u| < u_0$,

(a) The multiplicity of $E_0^{(\ell)}$ is equal to 1. (That is, the ground state is unique)
(b) $\Delta^{(\ell)} \geqslant c$ for all ℓ

Expected: On the chain with **odd** number of sites, Hamiltonian should also be gapped

Proved using method of **[Kennedy, Tasaki '92]**

Loop representation



Theorem [Tóth '93; Aizenman, Nachtergaele '94; U '13]

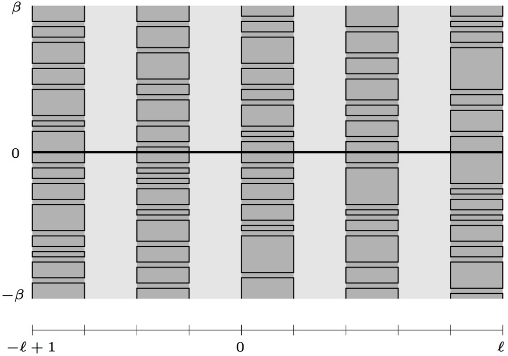
For the Hamiltonian with interactions $-uT_{x,x+1} - Q_{x,x+1}$, we have

(a) $\mathrm{Tr}\, \mathrm{e}^{-2\beta H_\ell} = \int \rho_u(\mathrm{d}\omega)\, n^{\mathcal{L}(\omega)}$

(b) $\mathrm{Tr}\, L_x^{\alpha,\alpha'} L_y^{\alpha,\alpha'}\, \mathrm{e}^{-2\beta H_\ell} = \int \rho_u(\mathrm{d}\omega)\, n^{\mathcal{L}(\omega)} \big(\mathbb{1}[x \overset{-}{\longleftrightarrow} y] - \mathbb{1}[x \overset{+}{\longleftrightarrow} y] \big)$

Heuristics

The loop measure is biased towards configurations with many loops.
Optimal way:



If these were typical configurations:

$$\mathbb{E}\Big[\mathbb{1}[0 \overset{-}{\longleftrightarrow} 1] - \mathbb{1}[0 \overset{+}{\longleftrightarrow} 1]\Big] = 1 > 0 = \mathbb{E}\Big[\mathbb{1}[-1 \overset{-}{\longleftrightarrow} 0] - \mathbb{1}[-1 \overset{+}{\longleftrightarrow} 0]\Big]$$

Method

- Introduce “contours”: those loops that are not short
- Method of cluster expansion. The difficulty is to prove convergence, as the cost of contours is purely entropic

The result is that correlations are equal to those of “optimal” configurations, up to a controlled error

It also gives exponential decay of spin-spin correlations

Proof of the gap 1/2

Method of **[Kennedy, Tasaki '92]**

Recall that $Z_{\ell,\beta} = \mathrm{Tr}\, \mathrm{e}^{-2\beta H_\ell}$. From cluster expansions, there exists $n_0, u_0, c > 0$ (independent of ℓ, β) and C_ℓ (independent of β) such that for all $n \geqslant n_0$ and $|u| \leqslant u_0$, we have for all β that

$$\Big| E_0^{(\ell)} + \tfrac{1}{2\beta} \log Z_{\ell,\beta} \Big| \leqslant C_\ell \, \mathrm{e}^{-\beta c}$$

Proof of the gap 2/2

Let $m_i^{(\ell)}$: multiplicity of eigenvalue $E_i^{(\ell)}$

$$Z_{\ell,\beta} = \sum_{i \geqslant 0} m_i^{(\ell)} \, \mathrm{e}^{-2\beta E_i^{(\ell)}} = m_0^{(\ell)} \, \mathrm{e}^{-2\beta E_0^{(\ell)}} \left(1 + \sum_{i \geqslant 1} \frac{m_i^{(\ell)}}{m_0^{(\ell)}} \, \mathrm{e}^{-2\beta (E_i^{(\ell)} - E_0^{(\ell)})} \right)$$

Thus

$$-\tfrac{1}{2\beta} \log Z_{\ell,\beta} = E_0^{(\ell)} - \tfrac{1}{2\beta} \log m_0^{(\ell)} - \tfrac{1}{2\beta} R(\ell,\beta),$$

where

$$R(\ell,\beta) = \log \left(1 + \sum_{i \geqslant 1} \frac{m_i^{(\ell)}}{m_0^{(\ell)}} \, \mathrm{e}^{-2\beta (E_i^{(\ell)} - E_0^{(\ell)})} \right)$$

Observe that

$$\mathrm{e}^{-3\beta \Delta^{(\ell)}} \leqslant R(\ell,\beta) \leqslant C_\ell \, \mathrm{e}^{-2\beta \Delta^{(\ell)}}$$

Recalling that $\left| E_0^{(\ell)} + \frac{1}{2\beta} \log Z_{\ell,\beta} \right| \leqslant C_\ell \, \mathrm{e}^{-\beta c}$, we get that

- $m_1^{(\ell)} = 1$
- $\Delta^{(\ell)} > \frac{1}{3} c$ uniformly in ℓ, n

Conclusion

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Conclusion

- For large spins, we prove dimerisation in a spin chain with $O(n)$-invariant interactions
- Also: exponential decay of spin-spin correlations, and presence of a gap
- Results obtained without exact solution, and without the "frustration-free" property
- The family of quantum spin chains with $O(n)$-invariant interactions has a rich phase diagram!

THANK YOU!