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BULK & EDGE INDICES FOR 2D TOPOLOGICAL SYSTEMS

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EXAMPLE $\frac{1}{2}$(Bernevig-Zhang-Hughes) Model

On $\ell^2(\mathbb{Z}^2)$, two right shift op.: e_1, e_2 std. basis of $\mathbb{R}^2$

$$(R_j\psi)(x) \equiv \psi(x-e_j) \qquad (x\in\mathbb{Z}^2, j=1,2)$$

Define on $\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^2$, $a\in\mathbb{R}$, $\sigma_1,\sigma_2,\sigma_3$ Pauli mat.

$$H^{\frac{1}{2}BHZ} := a\mathbb{1}\otimes\sigma_3 + \left[R_1\otimes\tfrac{1}{2}(\sigma_3 - i\sigma_1) + R_2\otimes\tfrac{1}{2}(\sigma_3 - i\sigma_2) + h.c.\right]$$

$$= \left(a\mathbb{1} + \mathrm{Re}\{R_1+R_2\}\right)\otimes\sigma_3 + \mathrm{Im}\{R_1\}\otimes\sigma_1 + \mathrm{Im}\{R_2\}\otimes\sigma_2$$

$$\mathrm{Re}\{A\} \equiv \tfrac{1}{2}(A+A^*) \quad , \quad \mathrm{Im}\{A\} \equiv \tfrac{1}{2i}(A-A^*)$$

FACT: If $a \neq 0, \pm 2$, $\exists$ gap for $H^{\frac{1}{2}BHZ}$ about zero energy.

If $\exists$ gap, define Hall conductivity

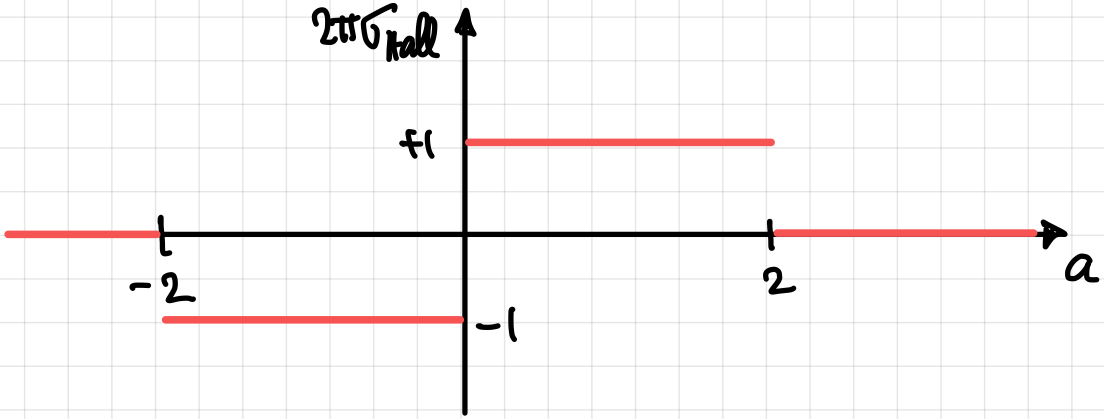
$$2\pi\,\sigma_{Hall} = 2\pi i \,\mathrm{tr}\left(P\left[[\Lambda_1, P], [\Lambda_2, P]\right] \right)$$

$$= \mathrm{index}\,(\underbrace{PUP + P^\perp}_{=:F}) \equiv \dim\ker F - \dim\ker F^* \quad \in \mathbb{Z}$$

w/ $P := \chi_{(-\infty,0)}(H)$ $\qquad U \equiv \exp(i \arg(X_1 + i X_2))$

$\Lambda_j \equiv \chi_{\mathbb{N}}(X_j) \qquad j = 1,2$

FACT:



This is the simplest non-trivial IQHE tight-binding model.

Now make this model Time-Reversal-Invariant (TRI):

Full Bernevig-Zhang-Hughes Model

On $\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, define

$$H^{BHZ} := (a\mathbb{1} + \mathrm{Re}\{R_1+R_2\})\otimes\sigma_3\otimes\mathbb{1} + \mathrm{Im}\{R_1\}\otimes\sigma_1\otimes\sigma_3 + \mathrm{Im}\{R_2\}\otimes\sigma_2\otimes\mathbb{1}$$

Note: H^{BHZ} is NOT block diagonal $\rightsquigarrow$ spin not conserved.

FACT: If $a\neq 0,\pm 2$, H^{BHZ} has a gap about zero.

Define Time-Reversal-Symmetry $\Theta := C\,\mathbb{1}\otimes\mathbb{1}\otimes(-i\sigma_2)$

C: complex-conj. of scalars

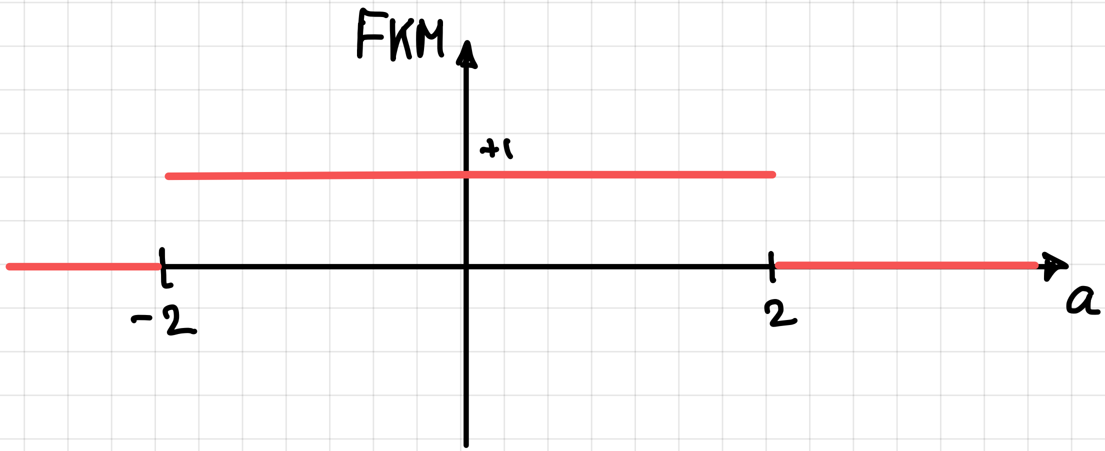
$\Theta^2 = -\mathbb{1}$ $\rightsquigarrow$ Fermionic TRS.

FACT: $[H^{BHZ}, \Theta] = 0$ $\rightsquigarrow$ $\sigma_{Hall}(H^{BHZ}) = 0$ $\forall\, a$.

FU-KANE-MELE (2007) have defined another topological index in this context — $\mathbb{Z}_2$-valued index counting whether edge states are unpaired.

$$\mathrm{FKM} := \mathrm{index}_2\,(\underbrace{PUP + P^\perp}_{\equiv F}) \equiv \left([\dim \ker F] \bmod 2 \right) \in \mathbb{Z}_2$$

$$P \equiv \chi_{(-\infty,0)}(H) \qquad\qquad U \equiv \exp(i \arg(X_1 + i X_2))$$



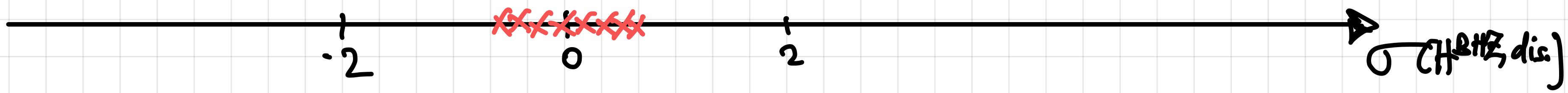
DISORDERED VERSION

Let $\{w(x)\}_{x \in \mathbb{Z}^2}$ be an IID seq. of R.V. w/ some nice prob. measure ρ.

$$H^{BHZ, dis.} := H^{BHZ} + w(X) \mathbb{1} \otimes \sigma_3 \otimes \mathbb{1}$$

__Effect:__ make the parameter "a" vary randomly across sites.

Depending on ρ, $H^{BHZ, dis.}$ will exhibit Anderson localization about zero.



Results about general models:

<u>Theorem(s)</u>: If H exhibits Anderson loc. about zero, $\mathbb{Z}$ and $\mathbb{Z}_2$ indices remain well-defined, obey a bulk-edge correspondence, and are independent of the choice of Fermi energy within the localized interval (= the mobility gap).

TECHNICAL DETAILS

DEF.: $A \in \mathcal{B}(\ell^2(\mathbb{Z}^d))$ is called weakly-local iff $A \in WLOC$ iff $\exists\, \nu \in \mathbb{N}: \quad \forall\, \alpha \in \mathbb{N},\ \exists\, C_\alpha \in (0,\infty):$

$$\|A_{xy}\| \leq C_\alpha (1+\|x-y\|)^{-\alpha} (1+\|x\|)^{+\nu}$$

For any interval $\Delta \subseteq \mathbb{R}$, let $B_1(\Delta)$ be the set of msrbl. f^n $f: \mathbb{R} \to \mathbb{C}$: ① f is const. above and below Δ.

② $\|f\|_\infty \leq 1$.

DEF.: $H = H^*$ has a deterministic mobility gap on Δ if ① $f(H) \in WLOC$ with estimates uniform as $f \in B_1(\Delta)$.
② $\sigma(H) \cap \Delta$ has finite degeneracy.

Note: $\chi_{(-\infty,0)} \in B_1(\Delta) \implies P \equiv \chi_{(-\infty,0)}(H) \in WLOC$ when H has a det. mob. gap on some $\Delta \ni 0$.

Thm.: (Aizenman–Graf, Graf–Elgart–Schenker, ...)

$P \in WLOC \implies F \equiv PUP + P^\perp$ is Fredholm.

Proof: $PUP + P^\perp$ is Fredholm iff $[P,U] \in \mathrm{Cpt}$.

Indeed, then $PU^*P + P^\perp$ is the parametrix:

$$
\begin{aligned}
\mathbb{1} - (PUP + P^\perp)(PU^*P + P^\perp) &= P - PUPU^*P \\
&= P(UU^* - UPU^*)P = PUP^\perp U^*P \\
&= [P,U]P^\perp U^*P
\end{aligned}
$$

But $[P, f(X)] \in \mathrm{Cpt}.$ whenever $P \in WLOC$ and

$$|f(x) - f(y)| \leq D \frac{\|x-y\|}{1+\|x\|}.$$

SULE basis (Simon et al...)

$\{\psi_n\}_n$ is a SULE basis for H on Δ

iff $\exists$ loc. centers $\{x_n\}_n \subseteq \mathbb{Z}^d$ and $\partial \in \mathbb{N} : \forall \alpha > 0$

$\exists$ $C_\alpha \in (0,\infty)$: $\quad \|\psi_n(x)\| \leq C_\alpha (1+\|x-x_n\|)^{-\alpha}(1+\|x_n\|)^{\partial}$

Lemma: If H has a det. mob. gap on Δ then

$\exists$ SULE basis.

STABILITY THEOREM

Let $\mu \in \Delta$ and $P_\mu := \chi_{(-\infty,\mu)}(H)$.

Thm.: If H has a det. mob. gap on Δ then

$$\Delta \ni \mu \longmapsto \mathrm{index}_{(2)}(P_\mu U P_\mu + P_\mu^\perp)$$

is constant.

Pf.: Let $\mu' > \mu$, $P := P_\mu$, $Q := P_{\mu'} - P$

$$\mathbb{P}U := PUP + P^\perp.$$

$$\mathbb{P}_{\mu'}U = PUP + QUQ + (P+Q)^\perp + PUQ + QUP$$

But PUQ is cpt.. Indeed:

$$|PUQ|^2 \equiv QU^*PUQ = Q(\underbrace{U^*PU - P}_{\in \mathcal{J}_3})Q$$

$$\Rightarrow \operatorname{index}_{(2)} \mathbb{P}_{\mu} U = \operatorname{index}_{(2)} PUP + QUQ + (P+Q)^{\perp}$$

$$\ker(PUP+QUQ+(P+Q)^{\perp}) \cong (\ker PUP) \oplus (\ker QUQ)$$

$$\cong (\ker \mathbb{P} U) \oplus (\ker \mathbb{Q} U)$$

But $\operatorname{index}_{(2)} A \oplus B = \operatorname{index}_{(2)} A + \operatorname{index}_{(2)} B \pmod 2$.

Hence W.T.S. $\operatorname{index}_{(2)} \mathbb{Q} U = 0$ whenever $\mathbb{Q}$ projects <u>within the mobility gap.</u>

Is $\mathbb{Q}U - \mathbb{1}$ compact? <u>NO</u>.

By the above, $\exists$ SULE basis for $\operatorname{im}(\mathbb{Q})$.

Define $V : \operatorname{im}(\mathbb{Q}) \to \operatorname{im}(\mathbb{Q})$

$$\psi_n \longmapsto \exp(i \arg((x_n)_1 + i (x_n)_2))$$

localization ctr.

V is unitary on $\operatorname{im}(\mathbb{Q}) \Rightarrow \operatorname{index}_{(2)}(\mathbb{Q} V \mathbb{Q} + \mathbb{Q}^{\perp}) = 0$.

Claim: $Q(U-V)Q \equiv Q(\exp(i\arg(X_1+iX_2))-\exp(i\arg((x_n)_1+i(x_n)_2)))Q$

is cpt.

Proof: ψ_n decays very quickly away from x_n anyway...

But $\mathrm{index}_{(2)}$ is stable under cpt. perturbations.

OPEN PROBLEM

① Define a new metric d on $\mathcal{B}(\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^2)$ s.t.

$$\mathcal{S} := \left\{ H = H^* \in \mathcal{B}(\ell^2(\mathbb{Z}^2)\otimes\mathbb{C}^N) \,\middle|\, PUP + P^\perp \text{ is Fredholm} \right\}$$

is an open set.

② Show that set of path-connected components $\pi_0(\mathcal{S})$ induced by d is bijective w/

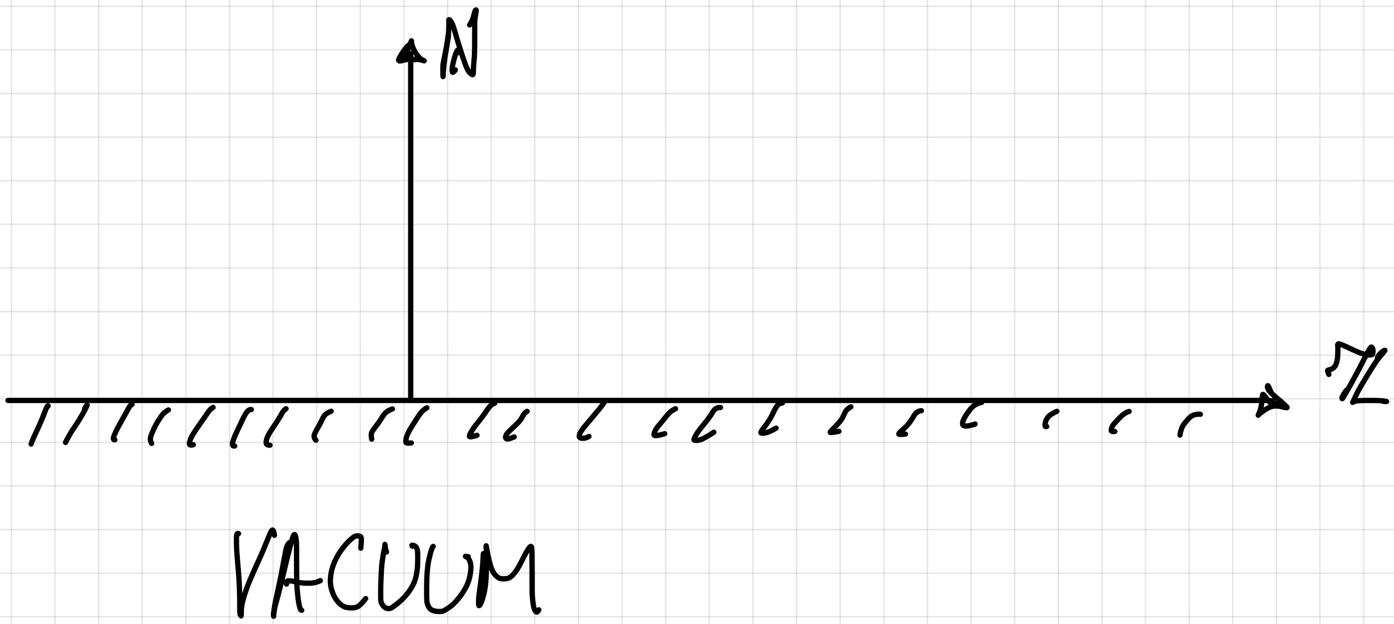
$\mathbb{Z}$ IQHE

$\mathbb{Z}$ TRI

via $\text{index}_{(2)}$.

③ Relation of d to transport coeff. / moments of X?

EDGE SYSTEMS



$$\hat{\mathcal{H}} := \ell^2(\mathbb{Z}\times\mathbb{N})\otimes\mathbb{C}^N$$

$$\iota : \ell^2(\mathbb{Z}\times\mathbb{N}) \hookrightarrow \ell^2(\mathbb{Z}^2)$$

↳ injection (extend by zero).

If H acts on $\mathcal{H}$, $\iota^* H \iota$ acts on $\hat{\mathcal{H}}$ (Dirichlet restriction).

As a rule, if H has a (spectral / mobility) gap, $\iota^* H \iota$ does NOT.

However, $\iota^* f(H)\iota - f(\iota^* H \iota) \longrightarrow 0$ (into bulk) for any smooth f.

$\Rightarrow$ If $\mathrm{supp}(f) \subseteq \Delta$, then $f(\iota^* H \iota)$ decays into bulk.

<u>Def.</u>: $\hat{H} = \hat{H}^* \in \mathcal{B}(\hat{\mathcal{H}})$ has a bulk-spec.-gap on Δ iff $(g^2 - g)(\hat{H})$ decays into bulk where

$g \approx$ smooth version of $\chi_{(-\infty,0)}$

with $\mathrm{supp}(g^2-g) \subseteq \Delta$.

Unfortunately for the mob. gap regime this cannot work, since then, H has eigenvalues in Δ which are centered everywhere, so

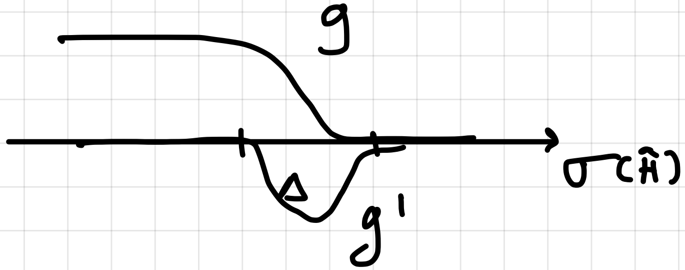
$$(g^2-g)(H) \neq 0$$ and does not even decay into bulk.

Intrinsic def. for the edge???

DEF.: $\hat{H} = \hat{H}^* \in \mathcal{B}(\hat{\mathcal{H}})$ has a bulk-mob.-gap on Δ iff $\exists$ bulk Hamiltonian $K \in \mathcal{B}(\mathcal{H})$ w/ mob. gap on Δ such that $\hat{H} - \iota^* K \iota \longrightarrow 0$ (into bulk).

This implies first def.

EDGE INDEX



IQHE: Edge hall conductivity

$$2\pi \hat{\sigma}_{Hall} = 2\pi \, tr\Big(g'(\hat{H}) \underbrace{i[\Lambda_1, \hat{H}]}_{\text{"velocity op."}} \Big)$$

$$= \text{index}\big(\Lambda_1 \exp(-2\pi i \, g(\hat{H}))\Lambda_1 + \Lambda_1^{\perp}\big) \in \mathbb{Z}$$

FKM: $\nexists$ trace formula.

Define $\text{index}_{(2)}\big(\Lambda_1 \exp(-2\pi i \, g(\hat{H}))\Lambda_1 + \Lambda_1^{\perp}\big)$

Relies on $[\Lambda_1, \mathbb{1} - e^{-2\pi i g(\hat{H})}] \in$ cpt.

$\Leftarrow \quad (g^2 - g)(\hat{H})$ decays in 2 direction

NOT true in mob. gap regime.

<u>Idea:</u> Remove localized bulk states from $g(\hat{H})$:

Since $\exists$ K w/ mob. gap on Δ :

$$\hat{H} - \iota^* K \iota \longrightarrow 0 \quad \text{into the bulk}$$

$$\hat{R} := \iota^* \chi_{\Delta^c}(K)\, \iota$$

In spec. gap regime, $\hat{R} = \mathbb{1}$ as $\chi_{\Delta^c}(K) = \mathbb{1}$.

Consider $\hat{F} := \Lambda_1 e^{-2\pi i \hat{R} g(\hat{H}) \hat{R}} \Lambda_1 + \Lambda_1^\perp$.

<u>Thm.:</u> $\hat{F}$ is Fredholm and its index is indep. of K as long as $\hat{H} - \iota^* K \iota \longrightarrow 0$ into bulk.

<u>Thm.:</u> $\text{index}_{(2)} F = \text{index}_{(2)} \hat{F}$ if $\hat{H} - \iota^* H \iota \longrightarrow 0$.