Continuous Edge Spectrum of Topological Insulators on the Lattice

Alex Bols
joint work with Albert Werner and Christopher Cedzich

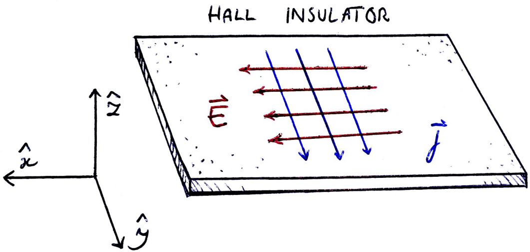
University of Copenhagen

22 juli 2022

Overview

- The quantum Hall effect
- The need for edge currents and absolutely continuous edge spectrum
- Two theorems (arXiv:2101.08603 [math-ph], arXiv:2203.05474 [math-ph])
- Sketch of proof

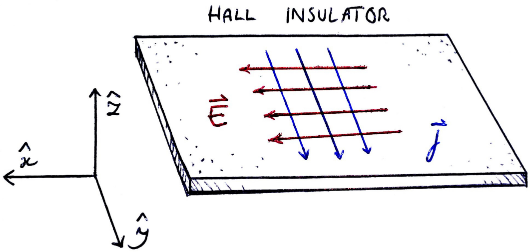
The bulk quantum Hall effect



- Bulk Hall current $\vec{j}_B$ in response to an electric field $\vec{E}$:

$$\vec{j}_B = \sigma \ \hat{z} \times \vec{E}.$$

The bulk quantum Hall effect



- Bulk Hall current $\vec{j}_B$ in response to an electric field $\vec{E}$:

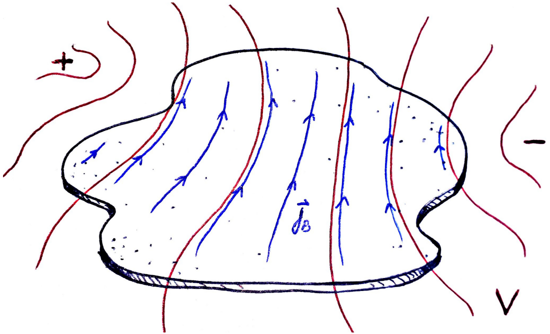
$$\vec{j}_B = \sigma \;\; \hat{z} \times \vec{E}.$$

- The *Hall conductance* σ of an electrical insulator is quantized:

$$\sigma \in \frac{e^2}{h}\mathbb{Z},$$

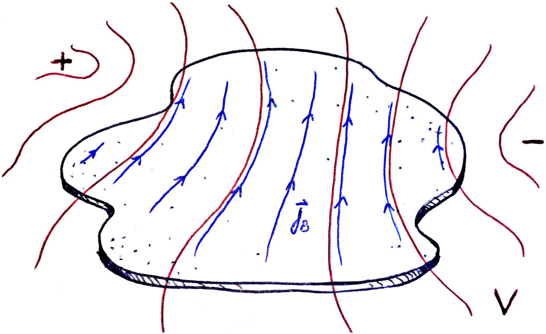
where e is the charge of an electron and h is Planck's constant.

A finite Hall insulator in a background potential



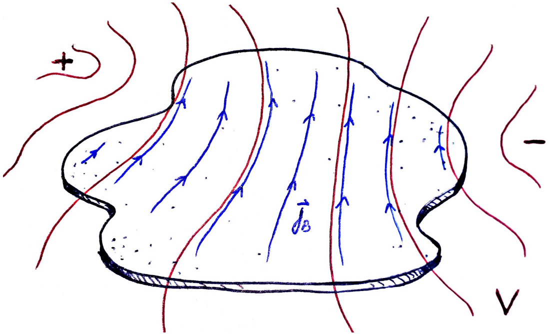
- The electrons in the Hall insulator are described by a single-particle Hamiltonian H with a gap $\Delta \subset \mathbb{R}$ in its spectrum.

A finite Hall insulator in a background potential



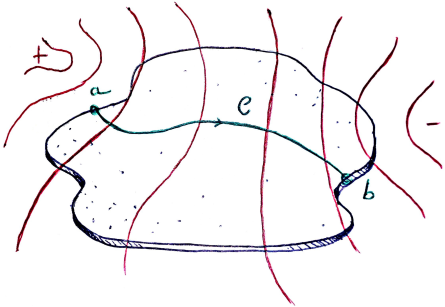
- The electrons in the Hall insulator are described by a single-particle Hamiltonian H with a gap $\Delta \subset \mathbb{R}$ in its spectrum.
- Apply a small external potential V and fill all single particle states of $H + V$ up to the Fermi energy $\mu \in \Delta$.

A finite Hall insulator in a background potential



- The electrons in the Hall insulator are described by a single-particle Hamiltonian H with a gap $\Delta \subset \mathbb{R}$ in its spectrum.
- Apply a small external potential V and fill all single particle states of $H + V$ up to the Fermi energy $\mu \in \Delta$.
- A bulk Hall current $\vec{j}_B$ flows along the equipotentials.

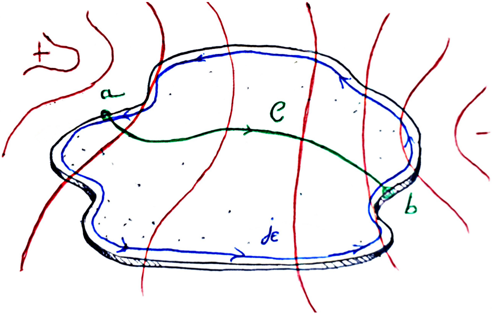
Necessity of Edge Currents



- No net current can flow through $\mathcal{C}$. The bulk current flowing through $\mathcal{C}$ is

$$\int_{\mathcal{C}} \mathrm{d}\hat{n} \cdot \vec{j}_B = -\sigma \int_{\mathcal{C}} \mathrm{d}\vec{l} \cdot \nabla V = \sigma(V(a) - V(b)).$$

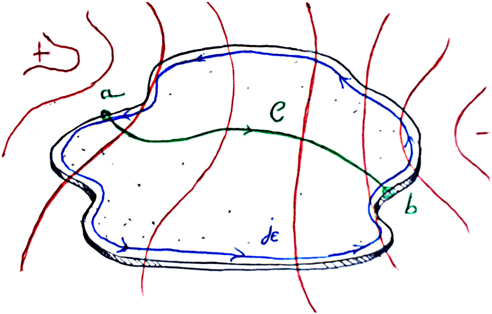
Necessity of Edge Currents



- Vanishing net current is achieved by positing the existence of an *edge current*

$$j_E(x) = \sigma V(x) + c.$$

Necessity of Edge Currents

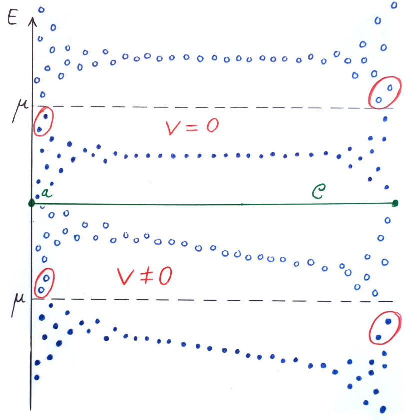


- Vanishing net current is achieved by positing the existence of an *edge current*

$$j_E(x) = \sigma V(x) + c.$$

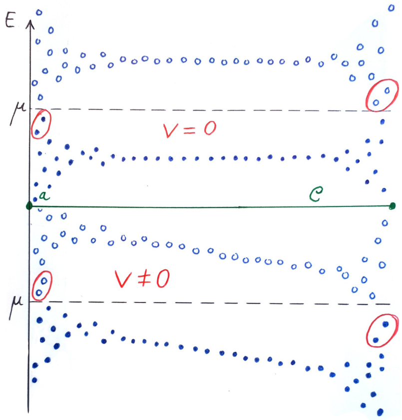
- By what mechanism could such an edge current arise?

Mechanism of the Edge Current



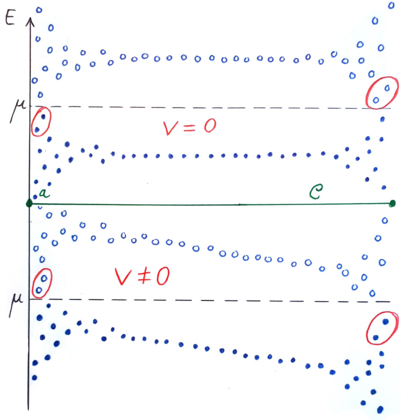
- The potential leads to a changed occupation of *edge modes*.

Mechanism of the Edge Current



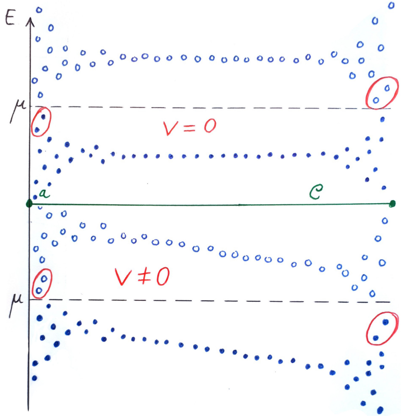
- The potential leads to a changed occupation of *edge modes*.
- The appropriate edge current arises if the edge modes carry a net current along the edge.

Mechanism of the Edge Current



- The potential leads to a changed occupation of *edge modes*.
- The appropriate edge current arises if the edge modes carry a net current along the edge.
- If a spectral window I of edge modes is occupied, then the electrons carry an edge current $\sigma\,|I|$.

Mechanism of the Edge Current



- The potential leads to a changed occupation of *edge modes*.
- The appropriate edge current arises if the edge modes carry a net current along the edge.
- If a spectral window I of edge modes is occupied, then the electrons carry an edge current $\sigma\,|I|$.
- An infinite edge supports ballistic motion along a line. This suggests absolutely continuous spectrum (cf. RAGE theorems).

Theorem for Hall insulators

- Let H be a single-particle local Hamiltonian on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ with spectral gap $\Delta \subset \mathbb{R}$, and let σ be the bulk Hall conductance of H filled up to the gap Δ.

Theorem for Hall insulators

- Let H be a single-particle local Hamiltonian on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ with spectral gap $\Delta \subset \mathbb{R}$, and let σ be the bulk Hall conductance of H filled up to the gap Δ.
- Concretely, for $\mu \in \Delta$ and $a \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ let

$$P = \chi_{\leq \mu}(H), \quad U_a = \mathrm{e}^{\mathrm{i}\arg(X-a)}$$

where $X = (X_1, X_2)$ is the vector of position operators on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ and put

$$\sigma = \frac{e^2}{h}\big(\dim\ker(U_a^* P U_a - P - \mathbb{1}) - \dim\ker(U_a^* P U_a - P + \mathbb{1})\big).$$

Theorem for Hall insulators

- Let H be a single-particle local Hamiltonian on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ with spectral gap $\Delta \subset \mathbb{R}$, and let σ be the bulk Hall conductance of H filled up to the gap Δ.
- Concretely, for $\mu \in \Delta$ and $a \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ let

$$P = \chi_{\leq \mu}(H), \quad U_a = \mathrm{e}^{\mathrm{i}\arg(X-a)}$$

where $X = (X_1, X_2)$ is the vector of position operators on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ and put

$$\sigma = \frac{e^2}{h}\big(\dim\ker(U_a^* P U_a - P - \mathbb{1}) - \dim\ker(U_a^* P U_a - P + \mathbb{1})\big).$$

- Let $\tilde{H}$ be the restriction of H to the half-space $\mathbb{N} \times \mathbb{Z}$.

Theorem for Hall insulators

- Let H be a single-particle local Hamiltonian on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ with spectral gap $\Delta \subset \mathbb{R}$, and let σ be the bulk Hall conductance of H filled up to the gap Δ.
- Concretely, for $\mu \in \Delta$ and $a \in \mathbb{R}^2 \setminus \mathbb{Z}^2$ let

$$P = \chi_{\leq \mu}(H), \quad U_a = \mathrm{e}^{\mathrm{i}\arg(X-a)}$$

where $X = (X_1, X_2)$ is the vector of position operators on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^n$ and put

$$\sigma = \frac{e^2}{h}\big(\dim\ker(U_a^* P U_a - P - \mathbb{1}) - \dim\ker(U_a^* P U_a - P + \mathbb{1})\big).$$

- Let $\tilde{H}$ be the restriction of H to the half-space $\mathbb{N} \times \mathbb{Z}$.

Theorem

If $\sigma \neq 0$, then $\Delta \subset \sigma_{a.c.}(\tilde{H})$.

Theorem for time-reversal invariant topological insulators

- Let τ be an on-site fermionic time-reversal action on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$, i.e. τ is an anti-unitary operator such that $\tau^2 = -\mathbb{1}$ and $[\tau, X_i \otimes \mathbb{1}] = 0$ for $i = 1, 2$.

Theorem for time-reversal invariant topological insulators

- Let τ be an on-site fermionic time-reversal action on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$, i.e. τ is an anti-unitary operator such that $\tau^2 = -\mathbb{1}$ and $[\tau, X_i \otimes \mathbb{1}] = 0$ for $i = 1, 2$.
- Assume now that H is *time-reversal symmetric*:

$$\tau H \tau^* = H. \tag{1}$$

It follows that the restriction $\tilde{H}$ is also time-reversal symmetric.

Theorem for time-reversal invariant topological insulators

- Let τ be an on-site fermionic time-reversal action on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$, i.e. τ is an anti-unitary operator such that $\tau^2 = -\mathbb{1}$ and $[\tau, X_i \otimes \mathbb{1}] = 0$ for $i = 1, 2$.
- Assume now that H is *time-reversal symmetric*:

$$\tau H \tau^* = H. \tag{1}$$

  It follows that the restriction $\tilde{H}$ is also time-reversal symmetric.
- Although $\sigma = 0$, we can consider a $\mathbb{Z}_2$-valued index

$$\mathrm{ind}_2 := \dim\ker\big(U_a^* P U_a - P - \mathbb{1}\big) \mod 2. \tag{2}$$

Theorem for time-reversal invariant topological insulators

- Let τ be an on-site fermionic time-reversal action on $l^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$, i.e. τ is an anti-unitary operator such that $\tau^2 = -\mathbb{1}$ and $[\tau, X_i \otimes \mathbb{1}] = 0$ for $i = 1, 2$.
- Assume now that H is *time-reversal symmetric*:

$$\tau H \tau^* = H. \tag{1}$$

  It follows that the restriction $\tilde{H}$ is also time-reversal symmetric.
- Although $\sigma = 0$, we can consider a $\mathbb{Z}_2$-valued index

$$\mathrm{ind}_2 := \dim \ker(U_a^* P U_a - P - \mathbb{1}) \mod 2. \tag{2}$$

Theorem

If $\mathrm{ind}_2 = 1$, *then* $\Delta \subset \sigma_{a.c.}(\tilde{H})$.

Sketch of proof

- We consider the *edge unitary*

$$\mathcal{U} := \exp\{2\pi \mathrm{i} f(\tilde{H})\}$$

where $f : R \to [0, 1]$ is a smooth function interpolating between 0 and 1 such that f' is supported on Δ.

Sketch of proof

- We consider the *edge unitary*

$$\mathcal{U} := \exp\{2\pi i f(\tilde{H})\}$$

  where $f : R \to [0,1]$ is a smooth function interpolating between 0 and 1 such that f' is supported on Δ.
- The unitary $\mathcal{U}$ acts non-trivially only on the edge states and therefore acts as the identity far away from the edge. Moreover, $\mathcal{U}$ is a smooth function of $\tilde{H}$ and therefore local.

Sketch of proof

- We consider the *edge unitary*

$$\mathcal{U} := \exp\{2\pi \mathrm{i} f(\tilde{H})\}$$

  where $f : R \to [0, 1]$ is a smooth function interpolating between 0 and 1 such that f' is supported on Δ.
- The unitary $\mathcal{U}$ acts non-trivially only on the edge states and therefore acts as the identity far away from the edge. Moreover, $\mathcal{U}$ is a smooth function of $\tilde{H}$ and therefore local.
- In fact one can show that $[\mathcal{U}, \Pi]$ is trace class with Π is projection on $\mathbb{N} \times \mathbb{N} \subset \mathbb{N} \times \mathbb{Z}$. In particular, $\mathcal{U}^* \Pi \mathcal{U} - \Pi$ is compact. (P. Elbau, G. Graf '02)

Sketch of proof

- We consider the *edge unitary*

$$\mathcal{U} := \exp\{2\pi i f(\tilde{H})\}$$

where $f : R \to [0,1]$ is a smooth function interpolating between 0 and 1 such that f' is supported on Δ.

- The unitary $\mathcal{U}$ acts non-trivially only on the edge states and therefore acts as the identity far away from the edge. Moreover, $\mathcal{U}$ is a smooth function of $\tilde{H}$ and therefore local.
- In fact one can show that $[\mathcal{U}, \Pi]$ is trace class with Π is projection on $\mathbb{N} \times \mathbb{N} \subset \mathbb{N} \times \mathbb{Z}$. In particular, $\mathcal{U}^* \Pi \mathcal{U} - \Pi$ is compact. (P. Elbau, G. Graf '02)
- We can associate to $\mathcal{U}$ a $\mathbb{Z}$-valued *edge index*

$$\mathrm{ind}^E(\mathcal{U}) := \dim\ker(\mathcal{U}^* \Pi \mathcal{U} - \Pi - \mathbb{1}) - \dim\ker(\mathcal{U}^* \Pi \mathcal{U} - \Pi + \mathbb{1})$$

which measures the amount of charge transported by $\mathcal{U}$ along the edge of the Hall insulator.

Sketch of proof

- If H and therefore $\tilde{H}$ are time-reversal symmetric then $\tau\mathcal{U}\tau^* = \mathcal{U}^*$. We can associate to quasi one-dimensional unitaries with this symmetry constraint a $\mathbb{Z}_2$-valued index

$$\mathrm{ind}_2^{E}(\mathcal{U}) := \dim\ker(\mathcal{U}^*\Pi\mathcal{U} - \Pi - \mathbb{1}) \mod 2. \tag{3}$$

Sketch of proof

- If H and therefore $\tilde{H}$ are time-reversal symmetric then $\tau\mathcal{U}\tau^* = \mathcal{U}^*$. We can associate to quasi one-dimensional unitaries with this symmetry constraint a $\mathbb{Z}_2$-valued index

$$\mathrm{ind}_2^E(\mathcal{U}) := \dim\ker(\mathcal{U}^*\Pi\mathcal{U} - \Pi - \mathbb{1}) \mod 2. \tag{3}$$

- In either case we have the *bulk-boundary correspondence* (J. Shapiro et al. : arXiv:1908.00910)

$$\sigma = \mathrm{ind}^E \tag{4}$$

and, if there is time-reversal symmetry,

$$\mathrm{ind}_2 = \mathrm{ind}_2^E. \tag{5}$$

(See also (A.B., J. Schenker, J. Shapiro : arXiv:2110.07068) for a more direct proof using explicit homotopies of Fredholm operators.)

Sketch of proof

- We now want to show that a non-trivial edge index implies a.c. spectrum for the edge unitary $\mathcal{U}$.

Sketch of proof

- We now want to show that a non-trivial edge index implies a.c. spectrum for the edge unitary $\mathcal{U}$.
- In the quantum Hall case this follows from a result of Joachim Asch, Olivier Bourget, Alain Joye (arXiv:1906.08181):

Sketch of proof

- We now want to show that a non-trivial edge index implies a.c. spectrum for the edge unitary $\mathcal{U}$.
- In the quantum Hall case this follows from a result of Joachim Asch, Olivier Bourget, Alain Joye (arXiv:1906.08181):

Theorem (J. Asch, O. Bourget, A. Joye)

If $\mathcal{U}$ is unitary and Π is a projection such that $[\mathcal{U}, \Pi]$ is trace class then ind^E is well defined and

$$\mathcal{U} = (S^{\mathrm{ind}^E} \oplus W) + T$$

where W has trivial index, S is a shift and T is trace-class.

Sketch of proof

- We now want to show that a non-trivial edge index implies a.c. spectrum for the edge unitary $\mathcal{U}$.
- In the quantum Hall case this follows from a result of Joachim Asch, Olivier Bourget, Alain Joye (arXiv:1906.08181):

Theorem (J. Asch, O. Bourget, A. Joye)

If $\mathcal{U}$ is unitary and Π is a projection such that $[\mathcal{U}, \Pi]$ is trace class then ind^E is well defined and

$$\mathcal{U} = (S^{\mathrm{ind}^E} \oplus W) + T$$

where W has trivial index, S is a shift and T is trace-class.

- This Theorem can be regarded as a sort of Wold decomposition of 1D unitaries.

Sketch of proof

- For the time-reversal symmetric case, we provide an analogous ‘symmetric Wold decomposition’.

Sketch of proof

- For the time-reversal symmetric case, we provide an analogous 'symmetric Wold decomposition'.

Theorem (C. Cedzich, A. B.)

If $\mathcal{U}$ is unitary, Π a projection such that $[\mathcal{U}, \Pi]$ is trace class and $\tau\mathcal{U}\tau^* = \mathcal{U}^*$ and $\tau\Pi\tau^* = \Pi$ then the edge index ind_2^E is well defined. Moreover, if $\mathrm{ind}_2^E = 1$ is non-trivial then

$$\mathcal{U} = (S \oplus S^* \oplus W) + T \tag{6}$$

where T is trace class.

Sketch of proof

- The shift has a.c. spectrum covering the whole unit circle and a.c. spectrum is stable under trace-class perturbations, so

$$\sigma_{a.c.}(\mathcal{U}) = U(1)$$

if the edge index is non-trivial.

Sketch of proof

- The shift has a.c. spectrum covering the whole unit circle and a.c. spectrum is stable under trace-class perturbations, so

$$\sigma_{a.c.}(\mathcal{U}) = U(1)$$

  if the edge index is non-trivial.
- Finally, by spectral mapping we pull the a.c. spectrum of $\mathcal{U}$ back to $\tilde{H}$ to obtain

$$\sigma_{a.c.}(\tilde{H}) \supset \Delta,$$

  concluding the proof.

Symmetric Wold decomposition and approximate edge channels

- Suppose the edge unitary $\mathcal{U}$ has non-trivial index $\mathrm{ind}_2^E(\mathcal{U}) = 1$. Following (arXiv:1611.04439), one can construct a unitary $\mathcal{W}$ which is a trace-class perturbation of $\mathcal{U}$ such that

$$\mathcal{W}\Pi\mathcal{W}^* - \Pi = \Pi_+ - \Pi_- \tag{7}$$

where Π_+ and Π_- are one-dimensional projections.

Symmetric Wold decomposition and approximate edge channels

- Suppose the edge unitary $\mathcal{U}$ has non-trivial index $\mathrm{ind}_2^E(\mathcal{U}) = 1$. Following (arXiv:1611.04439), one can construct a unitary $\mathcal{W}$ which is a trace-class perturbation of $\mathcal{U}$ such that

$$\mathcal{W}\Pi\mathcal{W}^* - \Pi = \Pi_+ - \Pi_- \tag{7}$$

where Π_+ and Π_- are one-dimensional projections.

- We then apply a Wold-like construction to Π_+ and Π_-. In particular, we set

$$\Pi_+^{(k)} := \mathrm{Ad}_{\mathcal{W}}^{(k-1)}(\Pi_+), \quad \Pi_-^{(k)} := \mathrm{Ad}_{\mathcal{W}^*}^{k}(\Pi_-). \tag{8}$$

Symmetric Wold decomposition and approximate edge channels

- We then show (inductively) that

Symmetric Wold decomposition and approximate edge channels

- We then show (inductively) that
  - These projections are mutually orthogonal

$$\Pi_{\sigma}^{(k)}\Pi_{\sigma'}^{(l)} = \delta_{k,l}\,\delta_{\sigma,\sigma'}\Pi_{\sigma}^{(k)}. \tag{9}$$

Symmetric Wold decomposition and approximate edge channels

- We then show (inductively) that
  - These projections are mutually orthogonal
$$\Pi_{\sigma}^{(k)}\Pi_{\sigma'}^{(l)} = \delta_{k,l}\,\delta_{\sigma,\sigma'}\Pi_{\sigma}^{(k)}. \tag{9}$$
  - For $k \leq 0$ we have $\Pi_{\sigma}^{(k)}\Pi = \Pi_{\sigma}^{(k)}$ and for $k \geq 1$ we have $\Pi_{\sigma}^{(k)}\Pi^{\perp} = \Pi_{\sigma}^{(k)}$.

Symmetric Wold decomposition and approximate edge channels

- We then show (inductively) that
  - These projections are mutually orthogonal
$$\Pi_\sigma^{(k)}\Pi_{\sigma'}^{(l)} = \delta_{k,l}\,\delta_{\sigma,\sigma'}\Pi_\sigma^{(k)}. \tag{9}$$
  - For $k \leq 0$ we have $\Pi_\sigma^{(k)}\Pi = \Pi_\sigma^{(k)}$ and for $k \geq 1$ we have $\Pi_\sigma^{(k)}\Pi^\perp = \Pi_\sigma^{(k)}$.
  - For each k, the projections $\Pi_\pm^{(k)}$ form a Kramers pair, i.e.
$$\tau\Pi_+^{(k)}\tau^* = \Pi_-^{(k)}. \tag{10}$$

Symmetric Wold decomposition and approximate edge channels

- We then show (inductively) that
  - These projections are mutually orthogonal

$$\Pi_\sigma^{(k)}\Pi_{\sigma'}^{(l)} = \delta_{k,l}\,\delta_{\sigma,\sigma'}\Pi_\sigma^{(k)}. \tag{9}$$

  - For $k \leq 0$ we have $\Pi_\sigma^{(k)}\Pi = \Pi_\sigma^{(k)}$ and for $k \geq 1$ we have $\Pi_\sigma^{(k)}\Pi^\perp = \Pi_\sigma^{(k)}$.
  - For each k, the projections $\Pi_\pm^{(k)}$ form a Kramers pair, i.e.

$$\tau\Pi_+^{(k)}\tau^* = \Pi_-^{(k)}. \tag{10}$$

- $\mathcal{W}$ acts as a right shift on the subspace spanned by the $\Pi_+^{(k)}$ and as a left-shift on the subspace spanned by the $\Pi_-^{(k)}$, proving the theorem.

Symmetric Wold decomposition and approximate edge channels

- We then show (inductively) that
  - These projections are mutually orthogonal
$$\Pi_\sigma^{(k)}\Pi_{\sigma'}^{(l)} = \delta_{k,l}\,\delta_{\sigma,\sigma'}\Pi_\sigma^{(k)}. \tag{9}$$
  - For $k \leq 0$ we have $\Pi_\sigma^{(k)}\Pi = \Pi_\sigma^{(k)}$ and for $k \geq 1$ we have $\Pi_\sigma^{(k)}\Pi^\perp = \Pi_\sigma^{(k)}$.
  - For each k, the projections $\Pi_\pm^{(k)}$ form a Kramers pair, i.e.
$$\tau\Pi_+^{(k)}\tau^* = \Pi_-^{(k)}. \tag{10}$$
- $\mathcal{W}$ acts as a right shift on the subspace spanned by the $\Pi_+^{(k)}$ and as a left-shift on the subspace spanned by the $\Pi_-^{(k)}$, proving the theorem.
- $\mathcal{U}$ is close to $\mathcal{W}$ far away from the cut. The spaces spanned by the $\Pi_+^(k)$ and the $\Pi_-^{(k)}$ are therefore approximate edge channels for the edge unitary $\mathcal{U}$.

Thank You!