Implementing Bogoliubov Transformations beyond the Shale–Stinespring Condition

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Outline

- Setting: family in Fock space $(\Psi_t)_{t\in\mathbb{R}} \subset \mathscr{F}$, generated by Hamiltonian H via $i\partial_t \Psi_t = H\Psi_t$
- Problem: formal expression H is ill–defined as operator
- (Standard) solution by cutoffs $\Lambda \in [0, \infty)$:

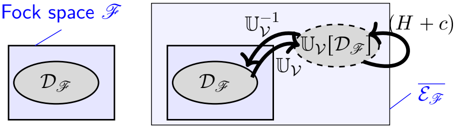
$$\widetilde{H}_{\text{cutoff}} := \lim_{\Lambda\to\infty} W_\Lambda^{-1}(H_\Lambda + c_\Lambda)W_\Lambda$$

  ($W_\Lambda : \mathscr{F} \to \mathscr{F}$: dressing operator, c_Λ: counterterm)
- When removing cutoffs, W formally leads **out of Fock space**
- We use **Fock space extensions** $\overline{\mathcal{E}_{\mathscr{F}}}$ (or $\widehat{\mathscr{H}}, \overline{\mathscr{F}}$) to directly define

$$\widetilde{H} := W^{-1}(H + c)W$$

- [L. 2020]: Construction of $\widetilde{H}$ using $\overline{\mathscr{F}}$ for W: Weyl trafo

Outline

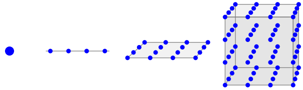


- We consider **quadratic Hamiltonian** H
- Can be diagonalized by **Bogoliubov transformation** $\mathcal{V}$ (algebraic modification of H)
- Goal: find **implementer** $W = \mathbb{U}_{\mathcal{V}}$ (operator), such that $\widetilde{H} := \mathbb{U}_{\mathcal{V}}^{-1}(H+c)\mathbb{U}_{\mathcal{V}}$ is diagonal
- $\exists$ impl. $\mathbb{U}_{\mathcal{V}} : \mathscr{F} \to \mathscr{F} \Leftrightarrow$ **Shale–Stinespring condition** holds
- **New result** [L. 2022]: $\exists$ ext. impl. $\mathbb{U}_{\mathcal{V}} : \mathscr{F} \supset \mathcal{D}_{\mathscr{F}} \to \overline{\mathcal{E}_{\mathscr{F}}}$ for bosonic $\mathcal{V}$ under very mild assumptions

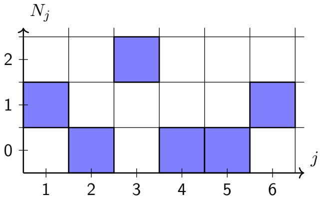
Fock Space Descriptions

- First description: by configuration space $\mathcal{Q}$:

$$q \in \mathcal{Q} = \mathcal{Q}(X) = \bigsqcup_{N=0}^{\infty} X^N$$



- $\mathscr{F} := L^2(\mathcal{Q})$ is called **Fock space**
- Separable $L^2(X)$ allows for basis $(e_j)_{j\in\mathbb{N}}$, so w.l.o.g. $X = \mathbb{N}$
- Second description: consider occupation number configs $\mathcal{Q}_{\mathrm{oc}} = \left\{(N_j)_{j\in\mathbb{N}} \mid \sum_j N_j < \infty\right\}$
- Then, $\mathscr{F} = L^2(\mathcal{Q}_{\mathrm{oc}})$



Quadratic Hamiltonians

- Creation/annihilation operators for $\boldsymbol{f} = (f_j)_{j\in\mathbb{N}} \in \ell^2$:

$$a^\dagger(\boldsymbol{f}) = \sum_j f_j a_j^\dagger \qquad a(\boldsymbol{f}) = \sum_j \overline{f_j} a_j$$

- Quadratic Hamiltonian ($+$: bosons, $-$: fermions) looks like:

$$H = \frac{1}{2} \sum_{j,k\in\mathbb{N}} \left(2h_{jk} a_j^\dagger a_k \mp k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k \right)$$

- H is element of *–algebra $\mathcal{A}$ (infinitely) generated by $a_j^\dagger, a_j$
- $a_j^\dagger, a_j$ satisfy **commutation relations** (CR)

$$[a_j^\dagger, a_k^\dagger]_\pm = [a_j, a_k]_\pm = 0, \qquad [a_j, a_k^\dagger]_\pm = \delta_{jk}$$

with $[A, B]_+ = AB - BA$ and $[A, B]_- = AB + BA$

Bogoliubov Transformations

- Consider the transformation $\mathcal{V}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$, $a^\sharp(\boldsymbol{f}) \mapsto b^\sharp(\boldsymbol{f})$ with

$$b^\dagger(\boldsymbol{f}) = a^\dagger(u\boldsymbol{f}) + a(v\overline{\boldsymbol{f}}), \qquad b(\boldsymbol{f}) = a(u\boldsymbol{f}) + a^\dagger(v\overline{\boldsymbol{f}})$$

where $u, v : \mathcal{D} \to \ell^2$ with $\mathcal{D} \subset \ell^2$: dense and $\boldsymbol{f} \in \mathcal{D}$

- Translating $a^\dagger(\boldsymbol{f}_1) + a(\overline{\boldsymbol{f}_2})$ into $\boldsymbol{F} = (\boldsymbol{f}_1, \boldsymbol{f}_2) \in \ell^2 \oplus \ell^2$ we identify $\mathcal{V}_{\mathcal{A}}$ with

$$\mathcal{V} = \begin{pmatrix} u & v \\ \overline{v} & \overline{u} \end{pmatrix}$$

- $\mathcal{V}$ (or $\mathcal{V}_{\mathcal{A}}$) is called **Bogoliubov transformation** if $b^\dagger, b$ still satisfy the (CR), and the same if $\mathcal{V}$ is replaced by $\mathcal{V}^*$

$$u^* u \mp v^T \overline{v} = 1 \qquad u^* v \mp v^T \overline{u} = 0$$
$$u u^* \mp v v^* = 1 \qquad u v^T \mp v u^T = 0$$

Diagonalization

- Action of $\mathcal{V}_{\mathcal{A}}$ on quadratic Hamiltonians is simple:

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} \left(2h_{jk} a_j^\dagger a_k \mp k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k \right)$$

- Consider $\widetilde{H} := \mathcal{V}_{\mathcal{A}} H$. If we identify H with the block matrix

$$A_H = \begin{pmatrix} h & \mp k \\ \overline{k} & \mp \overline{h} \end{pmatrix} \quad \text{then} \quad A_{\widetilde{H}} = \mathcal{V}^* A_H \mathcal{V}$$

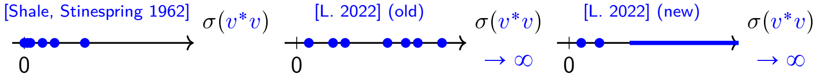
- Certain A_H allow for **diagonalization** (Stone's theorem), so

$$\exists \mathcal{V} : \quad A_{\widetilde{H}} = \begin{pmatrix} E & 0 \\ 0 & \mp \overline{E} \end{pmatrix} \quad \Rightarrow \quad \widetilde{H} = \sum_{j \in \mathbb{N}} E_j a_j^\dagger a_j$$

- In many cases, diagonalized $\widetilde{H}$ is densely defined and self–adjoint

Standard Implementation

- If $\exists \mathbb{U}_{\mathcal{V}}$: unitary with $\mathbb{U}_{\mathcal{V}}^* H \mathbb{U}_{\mathcal{V}} = \widetilde{H}$, then dynamics of H and $\widetilde{H}$ are equivalent ($\mathbb{U}_{\mathcal{V}}^* e^{-itH} \mathbb{U}_{\mathcal{V}} = e^{-it\widetilde{H}}$)
- We say that $\mathbb{U}_{\mathcal{V}}$ **implements** $\mathcal{V}$ and $\mathcal{V}$ is **implementable** iff
  $\exists \mathbb{U}_{\mathcal{V}}$ unitary : $\quad \mathbb{U}_{\mathcal{V}} a^\dagger(\boldsymbol{f}) \mathbb{U}_{\mathcal{V}}^* = b^\dagger(\boldsymbol{f}), \qquad \mathbb{U}_{\mathcal{V}} a(\boldsymbol{f}) \mathbb{U}_{\mathcal{V}}^* = b(\boldsymbol{f})$
- [Shale, Stinespring 1962]: $\mathcal{V} = \left(\begin{smallmatrix} u & v \\ v & u \end{smallmatrix}\right)$ implementable $\Leftrightarrow \operatorname{tr}(v^* v) < \infty$
- [Lill 2022] (old): $\mathcal{V}$ implementable in **extended sense** for $v^* v$ having **arbitrary discrete spectrum**
- [Lill 2022] (new): bosonic $\mathcal{V}$ implementable in **extended sense** for **generic** $v^* v$



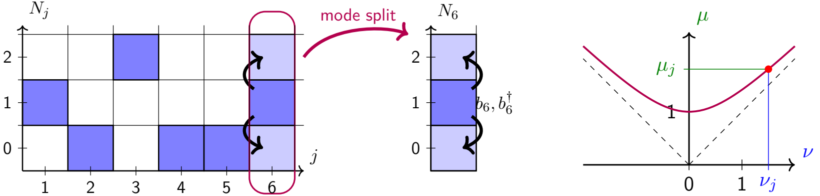
Standard Implementation

- Constructing $\mathbb{U}_{\mathcal{V}} : \mathscr{F} \to \mathscr{F}$ (unitary) works by finding **Bogoliubov vacuum** $\Omega_{\mathcal{V}} \in \mathscr{F}$ with

$$b(\boldsymbol{f})\Omega_{\mathcal{V}} = 0 \quad \forall \boldsymbol{f} \in \ell^2, \qquad (\mathbb{U}_{\mathcal{V}}\Omega := \Omega_{\mathcal{V}})$$

- Fact: If $\mathrm{tr}(v^* v) < \infty$, then $\exists$ bases $(\boldsymbol{f}_j)_{j\in\mathbb{N}}, (\boldsymbol{g}_j)_{j\in\mathbb{N}}$, s.th. (writing $b_j := b(\boldsymbol{f}_j), a_j := a(\boldsymbol{g}_j)$ and with $\mu_j, \nu_j \in \mathbb{R}, \mu_j^2 - \nu_j^2 = 1$):

$$b_j^\dagger = (\mu_j a_j^\dagger + \nu_j a_j)\Omega_{\mathcal{V}} = 0$$



Standard Implementation

- $\Omega_{\mathcal{V}}$ is product of pairs $K^{(2)} \in \ell^2 \otimes \ell^2$, normalized by $c_0 \in \mathbb{R}$, with $K^{(2)} = -\sum_j \frac{\nu_j}{2\mu_j} \boldsymbol{g}_j \otimes \boldsymbol{g}_j$:

$$\Omega_{\mathcal{V}}^{(N)} = \begin{cases} c_0 \frac{\sqrt{(2m!)}}{m!} (K^{(2)})^{\otimes_S m} & \text{if } N = 2m \\ 0 & \text{if } N: \text{ odd} \end{cases}$$

- $K^{(2)}$ is int. kernel of $\mathcal{O} : \ell^2 \to \ell^2$ with $2\mathcal{O}Ju\boldsymbol{f} = -vJ\boldsymbol{f}$
(Here, $J : \ell^2 \to \ell^2, J\boldsymbol{f} = \overline{\boldsymbol{f}}$ is the complex conjugation operator.)
- Now, $b(\boldsymbol{f})\Omega_{\mathcal{V}} = \left(a(u\boldsymbol{f}) + a^\dagger(v\overline{\boldsymbol{f}})\right)\Omega_{\mathcal{V}}$ with

$$a(u\boldsymbol{f})\Omega_{\mathcal{V}} \sim 2\sum_j (Ju\boldsymbol{f})_j K^{(2)}(j, j') \sim 2\mathcal{O}Ju\boldsymbol{f} \sim -vJ\boldsymbol{f}$$

- $a(u\boldsymbol{f})\Omega_{\mathcal{V}}$ cancels $a^\dagger(v\overline{\boldsymbol{f}})\Omega_{\mathcal{V}}$, so $b(\boldsymbol{f})\Omega_{\mathcal{V}} = 0$

Standard Implementation

- Knowing $\Omega_{\mathcal{V}}$, we can define $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\mathscr{F}} \to \mathscr{F}$ via

$$\begin{aligned}\mathbb{U}_{\mathcal{V}}(a^\dagger_{j_1} \dots a^\dagger_{j_N}\Omega) =& (\mathbb{U}_{\mathcal{V}} a^\dagger_{j_1} \mathbb{U}_{\mathcal{V}}^{-1})(\mathbb{U}_{\mathcal{V}} \dots \mathbb{U}_{\mathcal{V}}^{-1})(\mathbb{U}_{\mathcal{V}} a^\dagger_{j_N} \mathbb{U}_{\mathcal{V}}^{-1})(\mathbb{U}_{\mathcal{V}}\Omega)\\ :=& b^\dagger_{j_1} \dots b^\dagger_{j_N}\Omega_{\mathcal{V}}\end{aligned}$$

- $\mathcal{D}_{\mathscr{F}} := \operatorname{span}\{\Psi \in \mathscr{F} \mid \Psi = a^\dagger_{j_1} \dots a^\dagger_{j_N}\Omega\} \subset \mathscr{F}$ is dense
$\Rightarrow \mathbb{U}_{\mathcal{V}}$ can be defined on **all of** $\mathscr{F}$

Implementation beyond Shale-Stinespring

- Fact: $\|K^{(2)}\|_{\ell^2 \otimes \ell^2} < \infty \Leftrightarrow \mathrm{tr}(v^* v) < \infty$
- If $\sigma(v^* v)$ is arbitrary, even $(\boldsymbol{g}_j)_{j \in \mathbb{N}}$ may fail to exist
- However, $\mathcal{O}$ with $2\mathcal{O}Ju\boldsymbol{f} = -uJ\boldsymbol{f}$ still exists
- Use **Schwartz kernel theorem**: Let $\mathcal{D}$ be a nuclear space ("test functions") and $\mathcal{D}'$ its dual space ("distributions"). Then, any $\mathcal{O} : \mathcal{D} \to \mathcal{D}'$ has an integral kernel $K_{\mathcal{O}} \in \mathcal{D}' \otimes \mathcal{D}'$
- We choose:

$$\mathcal{D} := c_{00} = \{\boldsymbol{f} : \mathbb{N} \to \mathbb{C} \mid f_j \neq 0 \text{ finitely often}\}$$
$$\mathcal{E} := \mathcal{D}' = \{\mathbb{N} \to \mathbb{C}\}$$

so $\mathcal{D} \subset \ell^2 \subset \mathcal{E}$ is a **nuclear rigging**

- We have $\mathcal{O} : \mathcal{D} \to \mathcal{E}$, so $K_{\mathcal{O}} \in \mathcal{E} \otimes \mathcal{E}$ exists

Extended State Space

- For accommodating $\Omega_{\mathcal{V}}$, we define:

$$\mathcal{E}^{(N)} := \{\mathbb{N}^N \to \mathbb{C}\} = \mathcal{E}^{\otimes N} \qquad \mathcal{E}_{\mathscr{F}} := \bigoplus_{N=0}^{\infty} \mathcal{E}^{(N)}$$

- Thus, $K_{\mathcal{O}} \in \mathcal{E}^{(2)} = \mathcal{E} \otimes \mathcal{E}$ and $\Omega_{\mathcal{V}} \in \mathcal{E}_{\mathscr{F}}$ for

$$\Omega_{\mathcal{V}}^{(N)} = \begin{cases} c_0 \frac{\sqrt{(2m!)}}{m!} (K_{\mathcal{O}})^{\otimes_S m} & \text{if } N = 2m \\ 0 & \text{if } N : \text{ odd} \end{cases}$$

- Problem: annihilation operators may produce divergent sums
- "**Extended State Space**" $\overline{\mathcal{E}_{\mathscr{F}}}$ is a technical construct, rigorously accommodating (possibly divergent) sums

$$\Psi^{(N)}(j_1, \ldots, j_N) = \sum_{L \in \mathbb{N}_0} \sum_{j_{N+1}, \ldots, j_{N+L} \in \mathbb{N}} \Psi_{(L)}^{(N+L)}(j_1, \ldots, j_{N+L})$$

with $\Psi_{(L)}^{(N+L)} \in \mathcal{E}^{(N+L)}$

Main Result

Definition (Extended Implementation)

$\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\mathscr{F}} \to \overline{\mathcal{E}_{\mathscr{F}}}$ **implements** $\mathcal{V}$ **in the extended sense** on $\overline{\mathcal{E}_{\mathscr{F}}}$ iff

$$\mathbb{U}_{\mathcal{V}} a_j^\dagger \mathbb{U}_{\mathcal{V}}^{-1} \widetilde{\Psi} = b_j^\dagger \widetilde{\Psi}, \quad \mathbb{U}_{\mathcal{V}} a_j \mathbb{U}_{\mathcal{V}}^{-1} \widetilde{\Psi} = b_j \widetilde{\Psi} \quad \forall \widetilde{\Psi} \in \mathbb{U}_{\mathcal{V}}[\mathcal{D}_{\mathscr{F}}], j \in \mathbb{N}$$

Theorem ([L. 2022] Main Result)

*Any **bosonic** $\mathcal{V} = \begin{pmatrix} u & v \\ v & u \end{pmatrix}$ with $u, v : \mathcal{D} \to \ell^2$ is implementable in the ext. sense on $\overline{\mathcal{E}_{\mathscr{F}}}$.*

- [L. 2022]: For $\sigma(v^* v)$ being discrete, ext. impl. works for **bosons and fermions** on $\mathscr{F}$–extensions:
  - $\widehat{\mathscr{H}}$: Infinite tensor product space following [v. Neumann 1939], also used by [Blanchard 1969], [Fröhlich 1973], [Könenberg, Matte 2014]
  - $\overline{\mathscr{F}}$: Vector space related to $\overline{\mathcal{E}_{\mathscr{F}}}$
- Problem for fermions: $\mathcal{O}$ is unbounded, $\Omega_{\mathcal{V}}$ is more technical

Proof Outline and Comments

Lemma ($a_j^\dagger, a_j$ are well–defined)

For $j \in \mathbb{N}$, we have $a_j^\dagger : \overline{\mathcal{E}_{\mathscr{F}}} \to \overline{\mathcal{E}_{\mathscr{F}}}$ and $a_j : \overline{\mathcal{E}_{\mathscr{F}}} \to \overline{\mathcal{E}_{\mathscr{F}}}$

- Definition of $\mathbb{U}_{\mathcal{V}} : \mathcal{D}_{\mathscr{F}} \to \overline{\mathcal{E}_{\mathscr{F}}}$ via

$$\mathbb{U}_{\mathcal{V}}(a_{j_1}^\dagger \dots a_{j_N}^\dagger \Omega) := b_{j_1}^\dagger \dots b_{j_N}^\dagger \Omega_{\mathcal{V}}$$

Lemma (Condition for extended implementation)

If for $\mathbb{U}_{\mathcal{V}}$ as constructed above, we have $b_j \Omega_{\mathcal{V}} = 0$ ($\forall j \in \mathbb{N}$) and $\mathbb{U}_{\mathcal{V}}^{-1}$ exists, then $\mathbb{U}_{\mathcal{V}}$ implements $\mathcal{V}$ in the ext. sense

- $b_j \Omega_{\mathcal{V}} = 0$ follows from $2\mathcal{O}Ju\boldsymbol{f} = -uJ\boldsymbol{f}$ (same proof as for $\mathrm{tr}(v^* v) < \infty$)
- Existence of $\mathbb{U}_{\mathcal{V}}^{-1} \Leftrightarrow$ injectivity of $\mathbb{U}_{\mathcal{V}}$. Relies on $\|\mathcal{O}\| \leqslant 1/2$ and establishes the main theorem

What about the Counterterm c?

- Quadratic Hamiltonians come in different shapes: we used

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} \left(2h_{jk} a_j^\dagger a_k \mp k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k \right)$$

- With $[a_j, a_k^\dagger]_\pm = \delta_{jk}$, we may also write

$$H = \frac{1}{2} \sum_{j,k \in \mathbb{N}} \left(h_{jk} a_j^\dagger a_k \mp k_{jk} a_j^\dagger a_k^\dagger + \overline{k_{jk}} a_j a_k \mp \overline{h_{jk}} a_j a_k^\dagger \right) + c$$

with $c = \sum_j h_{jj} = \mathrm{tr}(h)$

- $c \notin \mathbb{C}$ may appear (“infinite constant”)
- But c is an infinite sum, so $c \in \overline{\mathcal{E}_{\mathscr{F}}}$

Examples for Ext.–Diagonalizable H

- **Quadratic Bosonic interactions** for $\varepsilon_{\boldsymbol{p}} \in \mathbb{R}$, $\kappa \in \mathbb{R}$

$$H = \frac{1}{2} \sum_{p \in \mathbb{Z}^3} \left(2(\varepsilon_{\boldsymbol{p}} + \kappa) a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + \kappa a_{\boldsymbol{p}}^\dagger a_{-\boldsymbol{p}}^\dagger + \kappa a_{\boldsymbol{p}} a_{-\boldsymbol{p}} \right)$$

- **BCS model** [Haag, 1962] (spin–$1/2$ fermions),
  $\varepsilon_{\boldsymbol{p}} \in \mathbb{R}$, $\tilde{\Delta}_{\boldsymbol{p}} \in \mathbb{C}$

$$H = \sum_{p \in \mathbb{Z}^3} \left(\varepsilon_{\boldsymbol{p}} a_{\boldsymbol{p},\uparrow}^\dagger a_{\boldsymbol{p},\uparrow} + \varepsilon_{\boldsymbol{p}} a_{\boldsymbol{p},\downarrow}^\dagger a_{\boldsymbol{p},\downarrow} - \tilde{\Delta}_{\boldsymbol{p}} a_{\boldsymbol{p},\uparrow}^\dagger a_{\boldsymbol{p},\downarrow}^\dagger + \overline{\tilde{\Delta}_{\boldsymbol{p}}} a_{\boldsymbol{p},\uparrow} a_{\boldsymbol{p},\downarrow} \right)$$

- **Extended field QED** model (fermionic),
  $\varepsilon_{\boldsymbol{p},\pm}(t) \in \mathbb{R}$, $f_{\boldsymbol{p}}(t) \in \mathbb{R}$

$$H(t) = \sum_{p \in \mathbb{Z}^3} \left(\varepsilon_{\boldsymbol{p},+}(t) a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + \varepsilon_{\boldsymbol{p},-}(t) b_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}} - f_{\boldsymbol{p}}(t) a_{\boldsymbol{p}}^\dagger b_{\boldsymbol{p}}^\dagger + f_{\boldsymbol{p}}(t) a_{\boldsymbol{p}} b_{\boldsymbol{p}} \right)$$

Thank you for your attention!