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A Quantum Detour: regularizing CED by means of QED

(Joint work with Z. Ammari, F. Hiroshima – arXiv:2202.05015)

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Electrodynamics of nonrelativistic charges

Classical charged particles interacting with the EM field

- Newton–Maxwell Equations:

(N-M)
$$\begin{cases} \dot{\mathbf{q}}_j = \frac{\mathbf{p}_j}{m_j} \\ \dot{\mathbf{p}}_j = m_j(\varrho_j * \mathbf{E})(\mathbf{q}_j) + \mathbf{p}_j \times (\varrho_j * \mathbf{B})(\mathbf{q}_j) - \nabla_j V(\mathbf{q}) \\ \\ \partial_t \mathbf{B}(\cdot) + \nabla \times \mathbf{E}(\cdot) = 0 \\ \partial_t \mathbf{E}(\cdot) - \nabla \times \mathbf{B}(\cdot) = -\sum_j \frac{\mathbf{p}_j}{m_j} \varrho_j(\cdot - \mathbf{q}_j) \\ \\ \nabla \cdot \mathbf{E}(\cdot) = \sum_j \varrho_j(\cdot - \mathbf{q}_j) \\ \nabla \cdot \mathbf{B}(\cdot) = 0 \end{cases}$$

Folklore: Disasters with (Almost) Point Charges

- Point Charges:

$$\varrho_{\forall j} = e_j \delta \quad \Longrightarrow \quad ↯$$

(electrostatic energy unbounded from below, atomic collapse by radiation)

- Charges with a small radius: [1]

$$\varrho_{\forall j} = e_j \mathbb{1}_{\left\{|\cdot| < \frac{2e_j^2}{3m_j}\right\}} \quad \Longrightarrow \quad ↯$$

(existence of runaway and non-causal solutions)

[1] E.J. Moniz, D.H. Sharp, *Phys. Rev. D* **15**(10), 1977.

Well-Posedness

- Global Well-Posedness ($V \in \mathscr{C}_{\mathrm{b}}^{2}$):

$\varrho_{\forall j}$ "regular enough" $\implies$ GWP on suitable Sobolev spaces for $\mathbf{E}$ and $\mathbf{B}$

$$\left(\varrho_{\forall j} \in H^1 \implies \text{GWP for } \mathbf{E} \in (H^{\frac{1}{2}})^{\times 3} \text{ and } \mathbf{B} \in (H^{\frac{1}{2}})^{\times 3}\right)$$

Quantized charged particles interacting with the Quantum EM field (Coulomb Gauge)

- Pauli-Fierz Hamiltonian:

$$\hat{H}_\hbar = \sum_{j=1}^{n} \frac{1}{2m_j}\big(\hat{p}_j - A_j(\hat{q}_j,\hat{a})\big)^2 + V(\hat{q}) + \hat{H}_{\mathrm{f}} \; ,$$

$$A_j(\hat{q}_j,\hat{a}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\epsilon_\lambda(k)}{\sqrt{2|k|}}\Big(\overline{\mathscr{F}\varrho_j}(k)\,\hat{a}_\lambda(k)e^{2\pi i k\cdot\hat{q}_j} + \mathscr{F}\varrho_j(k)\,\hat{a}^*_\lambda(k)e^{-2\pi i k\cdot\hat{q}_j}\Big)\mathrm{d}k \; ,$$

$$\hat{H}_{\mathrm{f}} = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} |k|\hat{a}^*_\lambda(k)\hat{a}_\lambda(k)\mathrm{d}k \; ,$$

$$[\hat{q}_j,\hat{p}_k] = i\hbar\delta_{jk} \; , \; [\hat{a}_\lambda(k),\hat{a}^*_\mu(p)] = \hbar\delta_{\lambda\mu}\delta(k-p) \; .$$

- Quantum Dynamics:

$$\varrho_\hbar(t) = e^{-i\frac{t}{\hbar}\hat{H}_\hbar}\varrho_\hbar e^{i\frac{t}{\hbar}\hat{H}_\hbar} \; .$$

Well-Posedness

- Global Well-Posedness ($V \in \mathscr{C}_{\mathrm{b}}^{2}$):

$$\varrho_{\forall j} \in \dot{H}^{-1} \cap \dot{H}^{\frac{1}{2}} \implies \hat{H}_{\hbar} \text{ is self-adjoint on } D(\hat{p}^2) \cap D(\hat{H}_{\mathrm{f}}) \ .$$

Remarks

- More singular Vs are allowed (*e.g.* Coulomb)
- Folklore is that point charges shall be admissible, however it is still mathematically an open problem (a renormalization is required)
- Atoms are stable, and no runaway or non-causal solutions are present

Bohr's Correspondence and Quantum Driven Classical Trajectories

Quantum Driven Classical GWP

Theorem 1 (Z. Ammari, MF, F. Hiroshima 2022)

$$\varrho_{\forall j} \in \dot{H}^{-1} \cap \dot{H}^{1-\sigma} \implies \text{(N-M) } \textit{GWP for } \mathbf{E} \in \left(H^{\sigma}\right)^{\times 3} \textit{ and } \mathbf{B} \in \left(H^{\sigma}\right)^{\times 3}$$

$$\boxed{0 \leq \sigma \leq \tfrac{1}{2}}$$

Proof of Theorem 1: Taking a Quantum Detour

$$
\begin{array}{ccc}
\varrho_\hbar[\mathbf{q}_0, \mathbf{p}_0, \mathbf{E}_0, \mathbf{B}_0] & \xmapsto{\; e^{-\frac{i}{\hbar} t \hat{H}_\hbar} \,(\,\cdot\,)\, e^{\frac{i}{\hbar} t \hat{H}_\hbar} \;} & \varrho_\hbar(t)[\mathbf{q}_0, \mathbf{p}_0, \mathbf{E}_0, \mathbf{B}_0] \\
\text{q-detour} \uparrow & & \downarrow \hbar \to 0 \\
u_0 = (\mathbf{q}_0, \mathbf{p}_0, \mathbf{E}_0, \mathbf{B}_0) & \xrightarrow{\;?\;\; \text{(N-M) flow}} & u_t = (\mathbf{q}_t, \mathbf{p}_t, \mathbf{E}_t, \mathbf{B}_t)
\end{array}
$$

Step 0

- Wigner Measures:

$$\varrho_\hbar \underset{\hbar \to 0}{\longrightarrow} \mathrm{d}\mu(u)$$

Step 1[2]

Theorem 2 (Z. Ammari, MF, F. Hiroshima 2022)

$$
\begin{array}{ccc}
\varrho_\hbar & \xmapsto{\; e^{-\frac{i}{\hbar}t\hat{H}_\hbar}\,(\,\cdot\,)\,e^{\frac{i}{\hbar}t\hat{H}_\hbar} \;} & \varrho_\hbar(t) \\
\Big\downarrow {\scriptstyle \hbar\to 0} & & \Big\downarrow {\scriptstyle \hbar\to 0} \\
\mu_0 & \xmapsto[\; \mathscr{L}_t^{(\mathrm{N-M})}\,(\,\cdot\,) \;]{} & \mu_t
\end{array}
$$

[2]For coherent states, Bohr's correspondence principle was established by A. Knowles, *PhD Thesis*, ETH Zürich, 2009.

Step 2

- *A priori* uniqueness:

$$\varrho_{\forall j} \in \dot{H}^{-1} \cap \dot{H}^{1-\sigma} \implies \text{There exists at most one } H^{\sigma}\text{-solution of (N-M)}$$

- *Liouville flow:* [3]

$$[\textit{A priori}\ !\ ] \wedge [\ ^{\exists}\mu_t = \mathscr{L}_t^{(\mathrm{N-M})}(\mu_0)] \implies {}^{\exists!}u_t \text{ sol. of (N-M) for } \mu_0\text{-a.a. } u_0$$

[3] C. Rouffort, *arXiv* 1809.01450, 2018.

Step 3

- *Saturating classical configurations via coherent states:*

$^{\forall}u_0 \ ^{\exists}\varrho_\hbar[u_0]$ (coherent state of minimal uncertainty): $\varrho_\hbar[u_0] \underset{\hbar \to 0}{\longrightarrow} \mathrm{d}\delta_{u_0}(u)$.

This concludes the proof of Theorem 1.

Outlook

Future Developments

- Application to other models : There are other models, perhaps less interesting physically (Fröhlich polaron), where the quantum-to-classical features appear even more transparently (classical instability vs. quantum stability, quantum-driven classical dynamics,...)
- Diamagnetic Inequality : classical $E_0(0) = E_0(\mathbf{A})$; quantum $E_\hbar(0) < E_\hbar(\mathbf{A})$
- Charges with small radii : Moniz-Sharp on solid mathematical grounds
- Point Charges : Solve the quantum obstructions to point particles, and define the classical point dynamics by taking the "quantum detour"

Thanks for the attention