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Session on

The ground state gap: existence, stability, and applications

Introduction

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Gapped ground state phases

- many-body systems at low temperatures are well-described by quantum lattice systems
- quantum phase transitions occur at $T = 0$ as parameters in the Hamiltonian vary
- generally one sees ‘critical phases’ and ‘gapped phases’
- gapped phases are often characterized by ‘topological’ order

Recent works that assume a gap above the ground state(s):

Quantization of conductance in gapped interacting systems, S. Bachmann, A. Bols, W. De Roeck, M. Fraas, arXiv:1707.06491

Quantization of Hall Conductance For Interacting Electrons on a Torus, M. B. Hastings, S. Michalakis., Commun. Math. Phys. **334**, 433–471 (2015)

The adiabatic theorem and linear response theory for extended quantum systems, S. Bachmann, W. De Roeck, M. Fraas, arXiv:1705.02838

Adiabatic currents for interacting electrons on a lattice, D. Monaco, S. Teufel, arXiv:1707.01852

A $\mathbb{Z}^2$-index of symmetry protected topological phases with time reversal symmetry for quantum spin chains, Y. Ogata, arXiv:1810.01045

Lieb-Schultz-Mattis type theorems for quantum spin chains without continuous symmetry, Y. Ogata, H. Tasaki, arXiv:1808.08740

Automorphic equivalence preserves the split property, A. Moon, arXiv:1903.00944

Recent proofs of a ground state gap:

Spectral gaps of frustration-free spin systems with boundary
M. Lemm, E. Mozgunov, J. Math. Phys. **60**, 051901 (2019)

The AKLT model on a hexagonal chain is gapped, M. Lemm, A. Sandvik, S. Yang, arXiv:1904.01043

A class of two-dimensional AKLT models with a gap, H. Abdul-Rahman, M. Lemm, A. Lucia, B. N., A. Young, arXiv:1901.09297

AKLT models on decorated square lattices are gapped N. Pomata, T.-C. Wei, arXiv:1905.01275

Gapped PVBS models for all species numbers and dimensions
M. Lemm, B. N., arXiv:1902.09678, Rev. Math. Phys. **9** (2019).

Finite-size criteria for spectral gaps in D-dimensional quantum spin systems M. Lemm, arXiv:1902.07141

Recent ‘perturbative’ results: ‘gap stability’

Topological quantum order: stability under local perturbations, S. Bravyi and M. Hastings and S. Michalakis, J. Math. Phys. **51**, 093512 (2010)

Stability of Frustration-Free Hamiltonians S. Michalakis, J.P. Zwolak, Commun. Math. Phys., **322**, 277–302 (2013)

Stability of Gapped Ground State Phases of Spins and Fermions in One Dimension, A. Moon, B. N., J. Math. Phys. 59, 091415 (2018)

Lieb-Robinson bounds, the spectral flow, and stability of the spectral gap for lattice fermion systems B. Nachtergaele, R. Sims, A. Young, arXiv:1705.08553, Proceedings QMATH13

M.B. Hastings. The stability of free Fermi hamiltonians. arXiv preprint arXiv:1706.02270 (2017)

Persistence of exponential decay and spectral gaps for interacting fermions, W. De Roeck, M. Salmhofer, arXiv:1712.00977

Lie-Schwinger block-diagonalization and gapped quantum chains J. Fröhlich, A. Pizzo, arXiv:1812.02457

Topics for discussion

- Introduction: The Bravyi-Hastings-Michalakis approach
- Daniel Ueltschi: Cluster expansion methods
- Alessandro Pizzo: Lie-Schwinger block-diagonalization method
- Wojciech De Roeck: Mobility gap versus spectral gap
- Martin Fraas: Split property in 2 dimensions
- More open problems: Simone Warzel, ...

The Bravyi-Hastings-Michalakis approach

1. Relative form bound $\implies$ gap stability.

Suppose

$$H(s) = H(0) + s\Phi, \quad H(0) \geq 0,$$

$$0 \in \mathrm{spec}(H(0)),\ (0, \gamma) \cap \mathrm{spec}(H(0)) = \emptyset,$$

and suppose there exist $\alpha \geq 0, \beta \in [0, 1)$such that

$$|\langle \psi, \Phi\psi \rangle| \leq \beta \langle \psi, H(0)\psi \rangle + \alpha \|\psi\|^2, \text{ for all } \psi$$

Then $\inf \mathrm{spec}(H_s) \in [-\alpha, \alpha]$ and

$$(|s|\alpha, (1 - |s|\beta)\gamma - |s|\alpha) \cap \mathrm{spec}(H(s)) = \emptyset.$$

and ‘ground states’ $\in [-|s|\alpha, +|s|\alpha]$.

2. System on ν-dimensional lattice $\Lambda \subset \Gamma$ with Hamiltonian

$$H_\Lambda(0) = \sum_{x \in \Lambda} h_x, \quad \Phi = \sum_{\substack{x,n \\ b(x,n) \subset \Lambda}} \Phi(b(x,n)).$$

Let P_Λ denote the ground state projection of $H_\Lambda(0)$.

Theorem (Michalakis-Zwolak, CMP 2013)

Assume h_x is frustration-free, H_0 has gap $\gamma > 0$, $\Phi(b(x,n))P_{b(x,n)} = 0$, for all x, n, and there is $M > 0$ be such that

$$\sum_{n \geq 1} n^\nu \|\Phi(b_x(n))\| \leq M, \text{ for all } x.$$

Then

$$|\langle \psi, \Phi\psi \rangle| \leq 3^\nu M \gamma^{-1} \langle \psi, H_\Lambda(0)\psi \rangle, \text{ for all } \psi \in \mathcal{H}_\Lambda.$$

This proves form boundedness for a class of perturbations.

3. Find conditions on the unperturbed model under which, after a suitable unitary transformation, a general class of perturbations can be brought into the form so that relative bound holds.

$$H_\Lambda(s) = \sum_{x\in\Lambda} h_x + s \sum_{\substack{x,n \\ b(x,n)\subset\Lambda}} \Phi(b(x,n)).$$

Assume h_x is frustration-free, H_0 has gap $\gamma > 0$, and an LTQO property, meaning there is a function Ω of suffiently fast decay such that, for all integers $0 \le k \le n \le L(\Lambda)$, $L(\Lambda) \to \infty$, and all observables $A \in \mathcal{A}_{b(x,k)}$,

$$\|P_{b(x,n)} A P_{b(x,n)} - \frac{\mathrm{Tr} P_\Lambda A}{\mathrm{Tr} P_\Lambda} P_{b(x,n)}\| \le |b(x,k)| \|A\| \Omega(n-k).$$

Slightly paraphrasing:

Theorem (Bravyi-Hastings-Michalakis-Zwolak, 2010–2013)

For $H_\Lambda(0)$ as above, if Φ is sufficiently short-range, there exists $s_0 > 0$ such that for $|s| < s_0$, there exists unitary $U(s)$ such that

$$U(s)^* H_\Lambda(s) U(s) = H_\Lambda(0) + s \sum_{\substack{x,n \\ b(x,n)\subset\Lambda}} \tilde{\Phi}(b(x,n)) + R_\Lambda(s) + E_\Lambda(s)\mathbb{1},$$

where $\tilde{\Phi}$ satisfies conditions of previous theorem and

$$\|R_\Lambda(s)\| \to 0 \text{ as } \Lambda \to \Gamma.$$

See forthcoming Part II of review article by N-Sims-Young.