# Critical Parameters of Loop and Bernoulli Percolation

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# Setting

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- Fix $\beta \in (0, \infty)$.
- To each $e \in E$ assign an independent Poisson point process of unit intensity placing crosses on $\{e\} \times [0, \beta)$.

This induces a random "permutation" $\sigma_\beta$.

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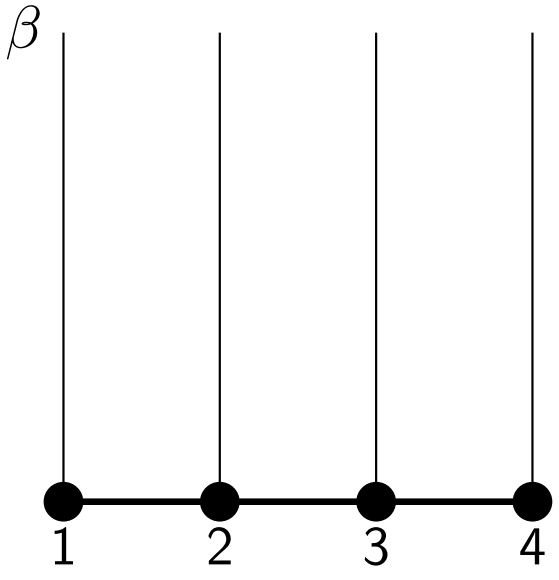

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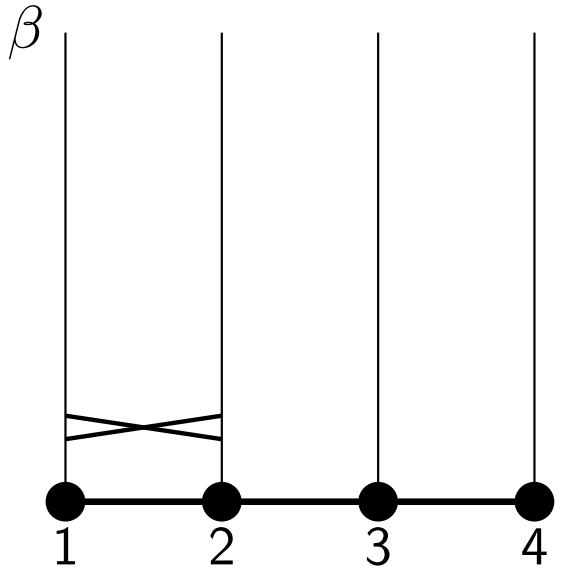

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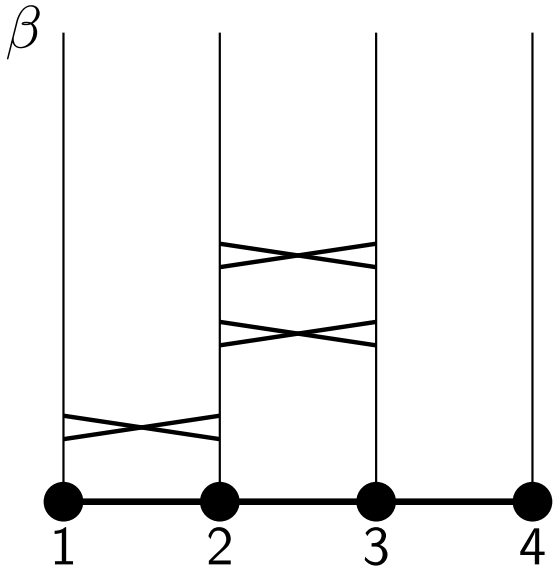

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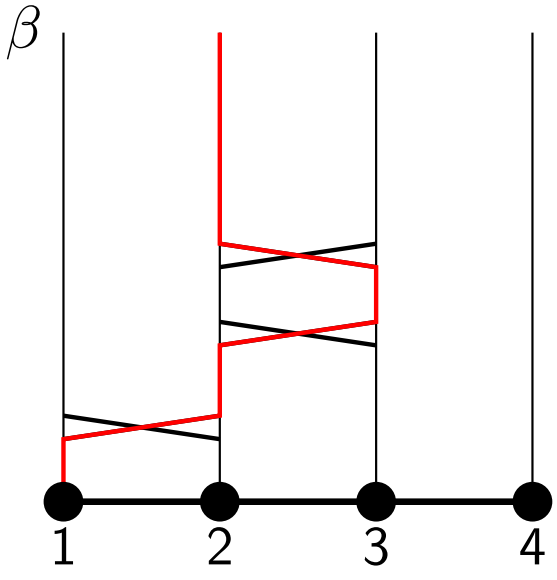

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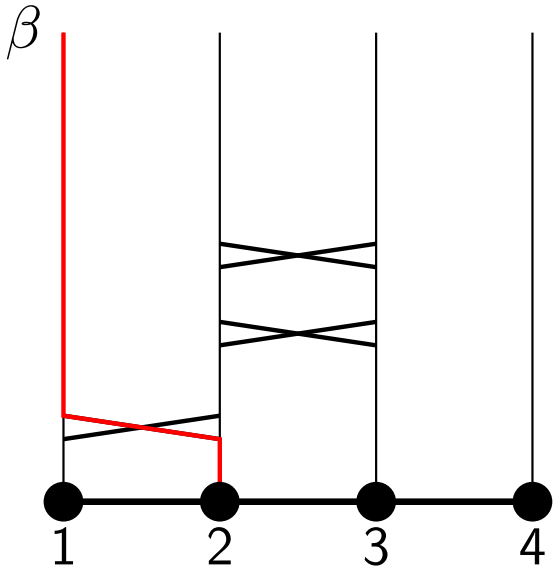

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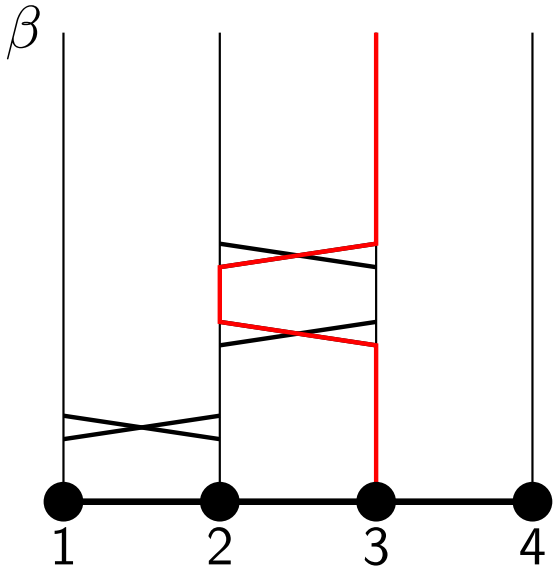

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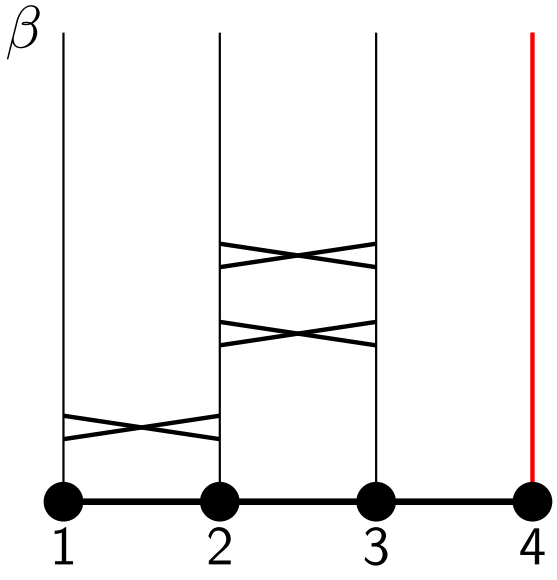

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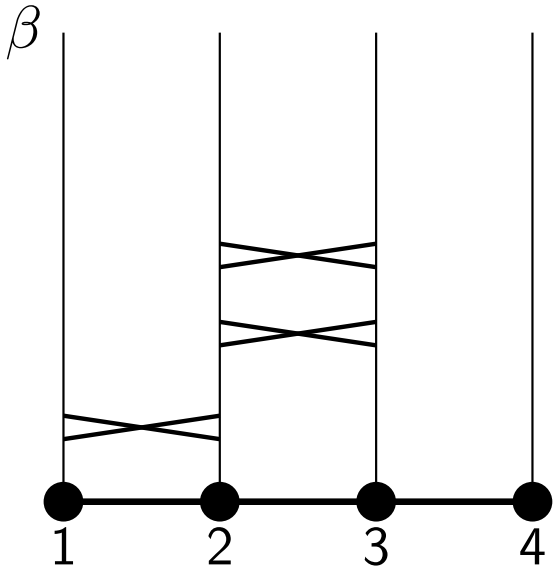


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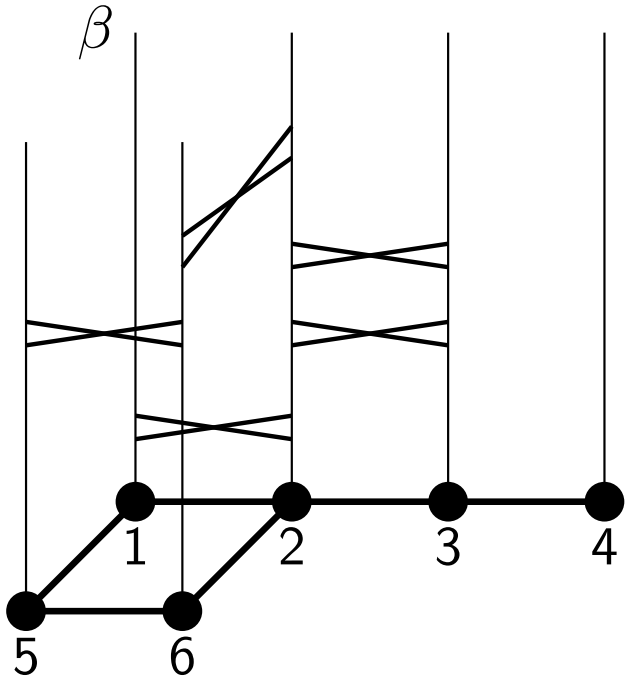



$\sigma_\beta = \tau_{12} \circ \tau_{23} \circ \tau_{23} \circ \tau_{56} \circ \tau_{26}$

# Motivation

- *Tóth '93:* Lower bound on pressure of the spin-$\frac{1}{2}$ quantum Heisenberg ferromagnet in terms of cycle lengths of random permutations.

  > *[...] the expected phase transition of the model is closely related to the appearance of an infinite cycle in the random stirring $\sigma_\beta$ of $\mathbb{Z}^d$, for $\beta$ sufficiently large.*

- *Aizenman, Nachtergaele '94:* Spin correlations of spin-$\frac{1}{2}$ quantum Heisenberg antiferromagnet in terms of "cycles" of a related model.
- *Ueltschi '13:* Extension of AN'94 to various other quantum spin models, including quantum XY and quantum ferromagnet.

# Progress so far (on appearance of large cycles)

Complete graph:

- *Schramm '05:* Explicitly calculated joint distribution of normalised cycle lengths. (In particular: Large cycles appear.)
- *Berestycki '10:* Direct proof for large cycles.
- *Alon, Kozma '12:* As above, using representation theory.

($d$-regular) trees:

- *Angel '03:* Large cycles appear. ($d \geq 5$)
- *Hammond '12,'13:* Large cycles appear ($d \geq 3$), more information about when they appear and that they stay.
- *Betz, Ehlert, Lees '18:* Large cycles appear (Galton-Watson trees with high offspring distribution).

Hypercube:

- *Kotecký, Miłoś, Ueltschi '16:* Large cycles appear.

Hamming graph:

- *Miłoś, Şengül '16:* Large cycles appear.

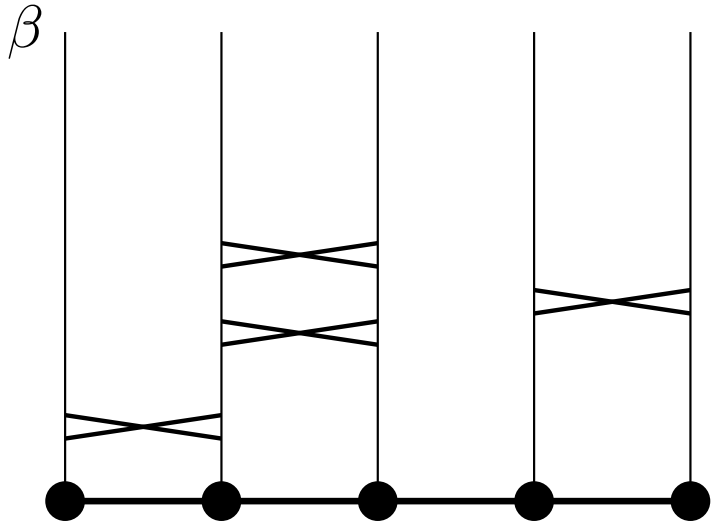
# Coupling with a percolation process

Can there be large cycles on $G = \mathbb{Z}$? I.e. is there a $\beta$ such that

$$\lim_{L\to\infty} \mathbb{P}_\beta(0 \text{ in cycle of size} > L) > 0 \text{ ?}$$

NO!

Couple our process to a Bernoulli percolation process by throwing away all edges without crosses:

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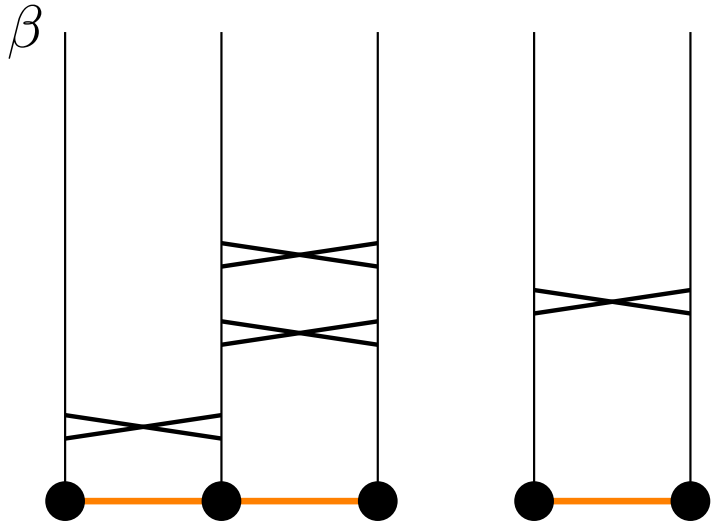


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# Percolation bound

Cycles always have to be subsets of percolation clusters, so we see that for general graphs $G$

$$\text{no infinite percolation cluster} \implies \text{no infinite cycles!}$$

Introducing

- $\beta_c^{\text{cycles}} := \inf\{\beta : \exists \text{ infinite cycle with positive probability}\}$,
- $\beta_c^{\text{perc}} := \inf\{\beta : \exists \text{ infinite percolation cluster with pos. prob.}\}$,

one has equivalently:

$$\beta_c^{\text{percolation}} \leq \beta_c^{\text{cycles}}.$$

# On sharpness of $\beta_c^{\text{percolation}} \leq \beta_c^{\text{cycles}}$

$\beta_c^{\text{percolation}} < \beta_c^{\text{cycles}}$ iff there is a $\beta$ such that

- $\exists$ an infinite percolation cluster with positive probability AND
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YES on $d$-regular trees (by Hammond '12), and in fact on all graphs of bounded vertex degree, e.g. $\mathbb{Z}^d$:

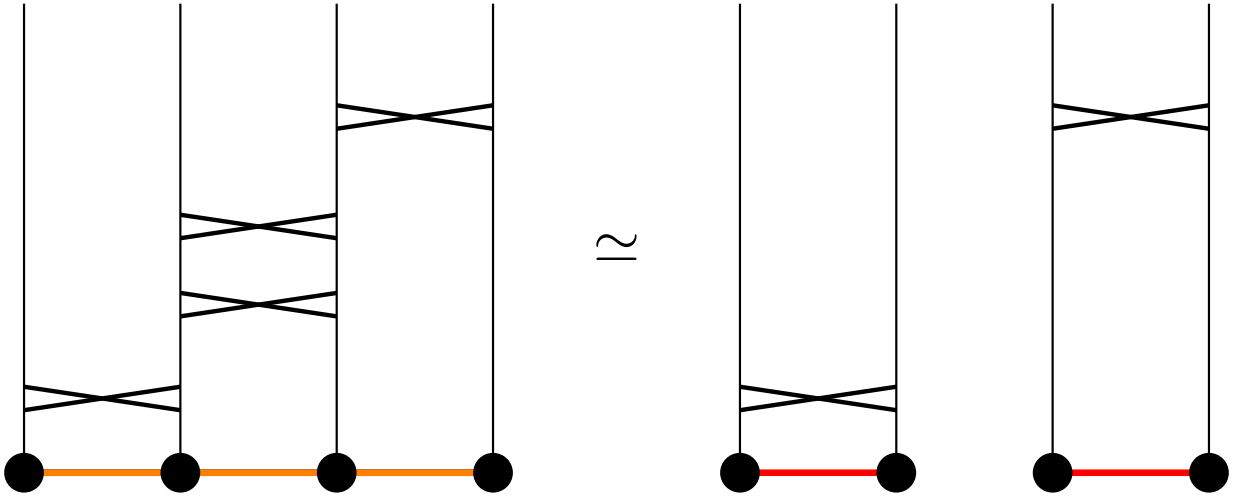
**Theorem (Mühlbacher '19)**

*On graphs of uniformly bounded vertex degree one has*

$$\beta_c^{\text{percolation}} < \beta_c^{\text{cycles}}.$$

# Key idea of the proof

Percolation bound is too generous:



So remove occurrences of such double crosses. If there are "enough", this will split up infinite clusters into finite ones.

# Where is the problem?

Naïvely: Take $\beta = \beta_c^{\text{percolation}} + \varepsilon$ and note that
$\mathbb{P}(e \text{ has double cross}) > 0$.
$\Rightarrow$ we throw away a positive fraction of edges
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Solution: Show that double crosses dominate a product measure.

# Conclusion

- Introduced the interchange process.
- Bounded the critical parameter for existence of infinite cycles in terms of percolation.
- Original contribution: Showed that, in contrast to graphs of diverging vertex degree, this bound is not sharp on graphs of bounded degree, e.g. $\mathbb{Z}^d$.

Thank you for your attention!