Critical Parameters of Loop and Bernoulli Percolation

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Setting

- Fix a graph $G = (V, E)$.
- Fix $\beta \in (0, \infty)$.
- To each $e \in E$ assign an independent Poisson point process of unit intensity placing crosses on $\{e\} \times [0, \beta)$.

This induces a random “permutation” $\sigma_\beta$. 

[Diagram: Points 1, 2, 3, 4 connected by lines]
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![Diagram showing the relationship between $\beta$, $e$, and the permutation $\sigma_\beta$.]
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![Diagram showing four points on a line with crosses at various positions, labeled 1, 2, 3, 4, and a vertical line labeled $\beta$.]
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\sigma_\beta = \tau_{12} \circ \tau_{23} \circ \tau_{23} \circ \tau_{56} \circ \tau_{26}
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Motivation

- **Tóth ’93**: Lower bound on pressure of the spin-$\frac{1}{2}$ quantum Heisenberg ferromagnet in terms of cycle lengths of random permutations.

  [...] the expected phase transition of the model is closely related to the appearance of an infinite cycle in the random stirring $\sigma_\beta$ of $\mathbb{Z}^d$, for $\beta$ sufficiently large.

- **Aizenman, Nachtergaele ’94**: Spin correlations of spin-$\frac{1}{2}$ quantum Heisenberg antiferromagnet in terms of “cycles” of a related model.

- **Ueltschi ’13**: Extension of AN’94 to various other quantum spin models, including quantum XY and quantum ferromagnet.
Progress so far (on appearance of large cycles)

Complete graph:

- *Schramm '05*: Explicitly calculated joint distribution of normalised cycle lengths. (In particular: Large cycles appear.)
- *Berestycki '10*: Direct proof for large cycles.
- *Alon, Kozma '12*: As above, using representation theory.

(d-regular) trees:

- *Angel '03*: Large cycles appear. ($d \geq 5$)
- *Hammond '12, '13*: Large cycles appear ($d \geq 3$), more information about when they appear and that they stay.
- *Betz, Ehlert, Lees '18*: Large cycles appear (Galton-Watson trees with high offspring distribution).

Hypercube:

- *Kotecký, Miłoś, Ueltschi '16*: Large cycles appear.

Hamming graph:

Coupling with a percolation process

Can there be large cycles on $G = \mathbb{Z}$? I.e. is there a $\beta$ such that

$$\lim_{L \to \infty} \mathbb{P}_\beta(0 \text{ in cycle of size } > L) > 0?$$

NO!

Couple our process to a Bernoulli percolation process by throwing away all edges without crosses:

Cycles have to be subsets of percolation clusters!
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Percolation bound

Cycles always have to be subsets of percolation clusters, so we see that for general graphs $G$

$$\text{no infinite percolation cluster} \implies \text{no infinite cycles!}$$

Introducing

- $\beta_c^{\text{cycles}} := \inf\{\beta : \exists \text{ infinite cycle with positive probability}\}$,
- $\beta_c^{\text{perc}} := \inf\{\beta : \exists \text{ infinite percolation cluster with pos. prob.}\}$,

one has equivalently:

$$\beta_c^{\text{percolation}} \leq \beta_c^{\text{cycles}}.$$
On sharpness of $\beta^\text{percolation}_c \leq \beta^\text{cycles}_c$

$\beta^\text{percolation}_c < \beta^\text{cycles}_c$ iff there is a $\beta$ such that

- there exists an infinite percolation cluster with positive probability
- there does not exist an infinite cycle almost surely.

Is there such a $\beta$?
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$\beta_c^{\text{percolation}} < \beta_c^{\text{cycles}}$ iff there is a $\beta$ such that

$\exists$ an infinite percolation cluster with positive probability AND

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Is there such a $\beta$?

NO on the complete graph (Schramm ’05), the Hamming graph (Miłoś, Şengül ’16), and on the hypercube (Kotecký, Miłoś, Ueltschi ’16).
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**YES** on $d$-regular trees (by Hammond '12), and in fact on all graphs of bounded vertex degree, e.g. $\mathbb{Z}^d$:

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**Theorem (Mühlbacher '19)**

On graphs of uniformly bounded vertex degree one has

$$\beta^\text{percolation}_c < \beta^\text{cycles}_c.$$
Key idea of the proof

Percolation bound is too generous:

So remove occurrences of such double crosses. If there are “enough”, this will split up infinite clusters into finite ones.
Where is the problem?

Naïvely: Take $\beta = \beta_c^{\text{percolation}} + \varepsilon$ and note that $\mathbb{P}(\text{e has double cross}) > 0$. 
$\Rightarrow$ we throw away a positive fraction of edges 
$\Rightarrow$ we are in the subcritical (percolation) regime 
$\Rightarrow$ there are no infinite cycles, since cycles are subsets of percolation clusters.
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Solution: Show that double crosses dominate a product measure.
Conclusion

- Introduced the interchange process.
- Bounded the critical parameter for existence of infinite cycles in terms of percolation.
- Original contribution: Showed that, in contrast to graphs of diverging vertex degree, this bound is not sharp on graphs of bounded degree, e.g. \( \mathbb{Z}^d \).

Thank you for your attention!