Braiding of Excitations in FQHE

Martin Fraas, Sven Bachmann, Alex Bols, Wojciech de Roeck,
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Setting — Hamiltonian

We consider a local Hamiltonian $H$ on a torus $\Gamma$ of $L^2$ sites, and let $P$ be its ground state projection.

Assumptions:

1. $H$ has a gap $\gamma > 0$ above the ground state,
2. $p$-fold ground state degeneracy, i.e. $\text{rank}(P) = p$,
3. (LTQO) for any $A$ with $l = L - \text{diam}(\text{supp}(A))$,

$$PAP = \frac{1}{p} \text{Tr}(PA)P + O(l^{-\infty}).$$

For concreteness,

$$H = \sum_{x \sim y, s, s'} g_{s,s'}(x, y) a_{x,s}^* a_{y,s'} + h.c. + \sum_{x, s} v (a_{x,s}^* a_{x,s})^2.$$

Remark: 1.-3. not known (but expected to happen) for $H$. 
Setting – Charge

For a region $\Omega$ we define $Q_{\Omega} = \sum_{x \in \Omega} Q_x$, the charge in region $\Omega$.

Assumptions:

1. $Q_x$ is supported on $\{x\}$,
2. $Q_x$ has integer spectrum,
3. Charge conservation, $[H, Q_\Gamma] = 0$.

For concreteness,

$$Q_x = \sum_s a^*_{x,s} a_{x,s}.$$  

Observation:

$[Q_\Gamma, P] = 0$,

but for $\Omega \subset \Gamma$

$[Q_\Omega, P] \neq 0$.  

Quantifying Fluctuations of Charge

For an observable \( A \) we define

\[
\bar{A} = \int_{-\infty}^{\infty} W(t) e^{iHt} A e^{-iHt} dt.
\]

Lemma (Hastings, Bachmann-Michalakis-Nachtergaele-Sims)

There exists \( W(t) \) decaying as \( te^{-ct/(\log t)^2} \) such that

\[
[\bar{A}, P] = 0.
\]

For a region \( \Omega \) we write \( \Omega^l = \{ x : \text{dist}(x, \Omega) < l \} \).

Corollary

Let \( \Omega \) be a region then there exists \( K \) supported on \( (\partial \Omega)^l \) such that

\[
\bar{Q} = Q - K + O(l^{-\infty}) \text{ and } K \text{ is a sum of local terms.}
\]
Contractible loops

Let $\Omega$ be a region such that $\partial \Omega$ is contractible, and let $U_{\partial \Omega} = \exp(2\pi i \bar{Q})$.

**Observation:** $U_{\partial \Omega}$ is supported on the boundary and preserves $P$. For any observable $X$ we have

$$[U_{\partial \Omega}, X] = O(\text{dist}(X, \partial \Omega)^{-\infty}), \quad [U_{\partial \Omega}, P] = 0.$$  

Moreover by $LTQO$

$$U_{\partial \Omega} P = P U_{\partial \Omega} P = e^{i \phi} P.$$  

We fix $\phi = 0$ by shifting $\bar{Q}$ by identity.
Non-contractible loops

For $\Omega$ with two boundaries $\partial_-$ and $\partial_+$ we have

$$\bar{Q}_\Omega = Q - K_- - K_+.$$  

Let $U_\pm = \exp(2\pi i (Q - K_\pm))$.

Lemma

*Suppose that* $\text{dist}(\partial_-, \partial_+) = O(L)$ *then*

$$[U_\pm, P] = O(L^{-\infty}).$$
Commutator on the ground state manifold

Let $U_{1,2}$ be unitaries corresponding to two complementary loops around the torus.

$$[U_1, P] = [U_2, P] = O(L^{-\infty}).$$

Theorem (Bachmann-Bols-de Roeck-F CMP 19 + in prep.)

There exists an integer $q$ such that

$$U_1 U_2 U_1^* U_2^* P = e^{2\pi i \frac{p}{q}} P + O(L^{-\infty}).$$

Moreover the Hall conductance $\sigma$ is equal to $p/q$. 
Open curves

1. Curve $\gamma$ connecting $x$ to $y$
2. Close it arbitrary to form $\Omega$
3. $\bar{Q}_\Omega = Q_\Omega - K$
4. Define
   $$U_{xy} = \exp(2\pi i (Q - K|_\gamma))$$

For a ground state $\psi$ put
$$\phi = U_{xy} \psi.$$ 

Observation: For any observable $X$, let $l = \text{dist}(X, \{x, y\})$ then

$$ (\phi, X\phi) = (\psi, X\psi) + O(l^{-\infty}). $$

$U_{xy}$ creates a pair of excitations at points $x, y$. 
Braiding Excitations

Moving excitations: If $U_{x\tilde{y}}$ creates excitation at $\tilde{y}$ then $U = U_{xy} U^*_{x\tilde{y}}$ moves this excitation from $y$ to $\tilde{y}$.

Braiding excitations: We have three excitations $x, y, z$ and we look at the effect of moving $z$ around $y$. Implemented by $V$.

\[ V\phi = VU_{xy}\psi = VU_{xy} V^* U^*_{xy} U_{xy} \psi = e^{2\pi i \frac{q}{p}} \phi + O(l^{-\infty}), \]
where $l$ is distance of $x, y$ to the loop of $z$. 