Prethermalization beyond high-frequency regime

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with my former master student Victor Verreet
=====> soon (?) on arxiv
(….. waiting for numerics)
Thermodynamic intuition

- local (many-body) Hamiltonians $H_1, H_2$ chain of length $L$

- Evolution after $t = n$ $U(n) \equiv U^n$, $U = e^{-iH_2}e^{-iH_1}$

\[
\lim_{t \to \infty} \langle O(t) \rangle \rightarrow \lim_{L} \text{tr}(O) \quad \text{for local } O
\]

- Possible obstruction:
  
  some local Ham $H_3$

\[
UH_3U^* = H_3 + O_{\text{local}}(\epsilon)
\]

\[
U^nH_3U^{-n} = H_3 + O_{\text{loc}}(n\epsilon)
\]
Obstruction...but usually also prethermalization

Possible obstruction:

some local Ham $H_3$

$$UH_3U^* = H_3 + O_{\text{local}}(\epsilon)$$

$$U^nH_3U^{-n} = H_3 + O_{\text{loc}}(n\epsilon)$$

- **Prethermal state**: “Quasi-stationary Noneq state” (Berges, Gasenzer, 2008-...)

- Only the obstruction is sometimes rigorous, not the thermalization and prethermalization (but Kos, Bertini, Prosen 2018)
Simplest example of obstruction: high frequency

\[ U = e^{-i\epsilon H_2} e^{-i\epsilon H_1} \approx e^{-i\epsilon (H_2 + H_1) + i\mathcal{O}_{\text{loc}}(\epsilon^2)} \]

Baker-Campbell-Hausdorf? No, converges only for \( ||H_i|| \sim 1 \)

Still, can construct \( H_3 = H_{\text{eff}} = \epsilon (H_1 + H_2) + \mathcal{O}_{\text{loc}}(\epsilon^2) \)

\[ U H_{\text{eff}} U^* = H_{\text{eff}} + \mathcal{O}_{\text{loc}}(e^{-1/\epsilon}) \]

Prethermalization up to exponential times!

(Magnus, .....D’Alesio et al,..... Rigourous 2017: Kuwahara et al, Abanin et al)
Motivation for this work

*Replica resummation of the Baker-Campbell-Hausdorff series*

(Vajna, Klobas, Prosen, Polkovnikov, PRL 2018)

Kicked many-body model: One-cycle unitary is

\[ U = e^{iJ \sum_i \sigma_i^x \sigma_{i+1}^x} e^{ih \sum_i (\cos(\theta) \sigma_i^z + \sin(\theta) \sigma_i^x)} \]

- \(|h| + |J| \ll 1 \quad \Rightarrow \quad \text{High-frequency regime}
  \text{Exponentially Slow heating & Prethermalization}

- \(|h| \ll |J| \sim 1 \quad \Rightarrow \quad \text{Moderate frequency but weak driving}
\[ U = e^{iJ} \sum_i \sigma_i^x \sigma_{i+1}^x \ e^{ih} \sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x) \]

- \(|h| + |J| \ll 1\) \(\Longrightarrow\) High-frequency regime
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- \(|h| \ll |J| \sim 1\) \(\Longrightarrow\) Moderate frequency but weak driving
  - Exponentially Slow heating !!

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Numerics and Replica Resummation give

\[ U^n \approx e^{inH_{\text{eff}}}, \quad n \leq e^{1/\sqrt{\epsilon}}, \quad \epsilon = h \text{ or } J \]
\[ U = e^{iJ \sum_i \sigma_i^x \sigma_{i+1}^x} e^{ih \sum_i (\cos(\theta) \sigma_i^z + \sin(\theta) \sigma_i^x)} \]

- \( |h| + |J| \ll 1 \) \( \Rightarrow \) High-frequency regime
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**Weak driving always gives exponentially slow heating?** No, in general
rate \( \sim (\text{driving})^2 \)

weakly interacting phonons or fermions

**Is there some simple special structure to these models?** Yes: this talk
A-posteriori motivation

\[ U = e^{iJ \sum_i \sigma_i^x \sigma_{i+1}^x} e^{i \sum_i (h_x \sigma_i^x + h_z \sigma_i^z)} \]

Numerics by Prosen 2007:

‘Minimal decay rate’ \( \Delta \) of local Ham

\[ U H_3 U^* = H_3 + \mathcal{O}_{\text{loc}}(\Delta) \]

White: \( \Delta \leq 10^{-6} \)

So it really matters whether \( h_x \) or \( h_z \) is small

special structure ?
Recall high-frequency regime

\[ H = J H_0 + h \cos(\Omega t) H_1 \quad J, h \ll \Omega \]

Many local events needed to absorb one photon of frequency \( \Omega \)

Dissipation only visible in order of PT \( \sim \Omega / J \)

heating rate \( \sim e^{-\Omega / J} \)

(Magnus, D’Alesio et al., Rigorous 2017: Kuwahara et al, Abanin et al)
Same logic: stability of doublons \( D\left(n_i = 2\right) \)

\[
H = \Omega \sum_i n_i(n_i - 1) + JH_1 \quad J \ll \Omega
\]

Many local events needed to provide D-energy \( \Omega \)

D – annihilation rate \( \sim e^{-\Omega/J} \)

(Sensarma et al, ….. Rigorous: Abanin et al, Else et al 2017)
Same logic: stability of doublons $D(n_i = 2)$

$$H = \Omega \sum_i n_i(n_i - 1) + J H_1 \quad J \ll \Omega$$

Many local events needed to provide $D$-energy

$D$ – annihilation rate $\sim e^{-\Omega/J}$

Wait... enough to have two distinct energy scales?

No, crucial property is: $\Omega \sum_i n_i(n_i - 1)$ can absorb only a discrete small set of energies locally.

- Sum of commuting local terms with integer gaps (as here)

- MBL systems (stability of MBL)
So: do we have “sums of commuting local terms with integer gaps”?

\[ U = e^{iJ \sum_i \sigma_i^x \sigma_{i+1}^x} e^{ih \sum_i (\cos(\theta) \sigma_i^z + \sin(\theta) \sigma_i^x)} \]

Yes, both terms have this property

\[ \sum_i (\cos(\theta) \sigma_i^z + \sin(\theta) \sigma_i^x) \]

\[ \sum_i \sigma_i^x \sigma_{i+1}^x \approx \text{doublons } D \]

To absorb doublon D, need to match frequency up to error of \( \mathcal{O}(h) \)

\[ \min_{m \in \mathbb{Z}} |n - m \frac{J}{2\pi}| \geq C h^n \]

Mechanism of exp. slow dissipation is there!
Our Theorem

\[ H(t) = J(t)D + hW(t), \quad t \in [0, 1] \]

Assumptions

\[ x := \frac{1}{2\pi} \int_{0}^{1} dt J(t) \quad \text{is “sufficiently Diophantine”} \]

\[ D \quad \text{is sum of commuting local terms with integer gaps} \]

periodicity \[ H(t) = H(t + 1) \]

Result

Take \( h \) small, time \( t < e^{C/h^{1/10}} \) and go to rotated frame:

\[ i\partial_t \Psi(t) = \~H(t)\Psi(t) \quad [\~H(t), D] = 0 \]

\( D \) is conserved \quad no heating

Can expect Prethermalization at \( D = \text{constant} \)
What means \[ x := \frac{1}{2\pi} \int_0^1 dt J(t) \] is “sufficiently Diophantine”

Recall: we need n'th order PT: \[ \min_{m \in \mathbb{Z}} |n - mx| \geq h^n \]

**Def:** \( x \) **is Diophantine:** \[ \min_{m \in \mathbb{Z}} |n - mx| \geq \frac{a(x)}{n^{-\tau}} \quad \tau > 1 \]

Most numbers are Diophantine:

\[ \text{size } \{x \in [0, 1]: a(x) > \epsilon \} \geq 1 - C\epsilon \]

**Our case:** \[ \frac{h}{a(x)} \] is the real small parameter
Example and Extension

Recall

\[ U = e^{iJ \sum_i \sigma_i^x \sigma_{i+1}^x} e^{i \sum_i (h_z \sigma_i^z + h_x \sigma_i^x)} \]

Assume now:

\[ h_z \ll h_x \sim J \sim 1 \]

instead of

\[ h_x^2 + h_z^2 \ll J^2 \sim 1 \]

New Diophantine condition:

\[ \min_{|n_1| + |n_2| + |n_3| \leq n} |n_1 J + n_2 h_x + 2 \pi n_3| \geq \frac{a}{n^{-\tau}} \]

Then: Both

\[ D = \sum_i \sigma_i^x \sigma_{i+1}^x \quad M = \sum_i \sigma_i^x \quad \text{quasi-conserved} \]

kinetically constrained model

Example

No local move possible
Both \( D = \sum_i \sigma_i^x \sigma_{i+1}^x \) \[ M = \sum_i \sigma_i^x \] quasi-conserved

General phenomenology:

- first order in \( h_z \): no spin flips at all
- First dissipation (spin flips) at time \( h_z^{-4} \)
- Prethermalization \( h_z^{-4} \leftrightarrow e^{1/h_z^{0.1}} \)
- Actually, even at order 4: dynamics is highly constrained
  
  further slowness depending on state
  
  droplet mass \( \sim \exp(\text{droplet length}) \)
  
  (magnetization, density of doublons)

- So even prethermalization might be very slow here -----'translation invariant (asymptotic) MBL'
Proof idea: KAM to exhibit conserved quantity

\[ H(t) = JD + hW(t), \quad t \in [0, 1] \]

- Goal (first order) for some \( W_d \) s.t. \([W_d, D] = 0\)

\[ e^{hA}(-i\partial_t + H)e^{-hA} = -i\partial_t + JD + hW_d + \mathcal{O}_{\text{loc}}(h^2) \]

- Suffices to solve linear ODE with periodic \( A \)

\[ -i\partial_t A + (W - W_d) + [A, D] = 0 \]

- Solution: (write \( \mathcal{N} \equiv iJ[D, \cdot] \))

\[ A(t) = e^{t\mathcal{N}}A(0) + \int_0^t ds e^{(t-s)\mathcal{N}}(W - W_d)(s) \]

- Imposing periodicity at time \( t=1 \):

\[ A(0) = \frac{1}{e^{\mathcal{N}} - 1} \]

(\[ \text{Resonance Denominator} \])
Conclusion

• Perturbative, rigorous view on slow heating in kicked Ising model

• We identified conditions for slow heating: small perturbations of Hamiltonians with commuting terms + Diophantine

• Not clear whether this indeed explains all the observed absence of heating in this model: numerics needed.