Prethermalization beyond high-frequency regime

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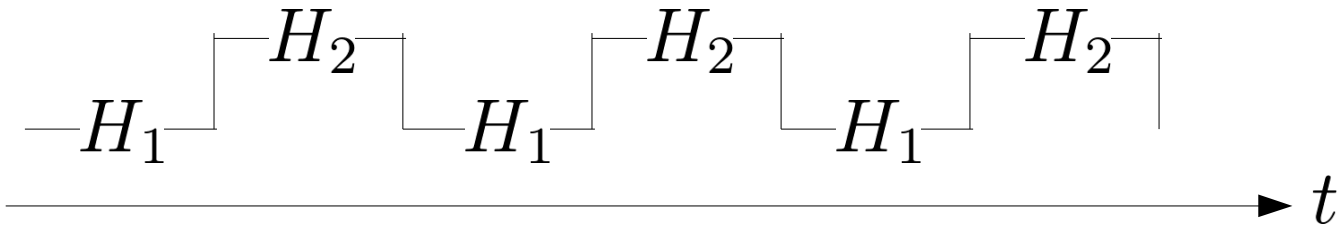
with my former master student **Victor Verreet**
====> soon (?) on arxiv

(….. waiting for numerics)

Thermodynamic intuition

- local (many-body) Hamiltonians H_1, H_2 chain of length L



- Evolution after $t = n$ $\quad U(n) \equiv U^n, \qquad U = e^{-iH_2} e^{-iH_1}$

…… should heat up to infinite temp.

$$\lim_L \langle O(t) \rangle \underset{t\to\infty}{\to} \lim_L \mathrm{tr}(O) \qquad \text{for local } O$$

- Possible obstruction:

some local Ham H_3

$$U H_3 U^* = H_3 + \mathcal{O}_{\mathrm{local}}(\epsilon)$$

$$U^n H_3 U^{-n} = H_3 + \mathcal{O}_{\mathrm{loc}}(n\epsilon)$$

Obstruction…but usually also prethermalization

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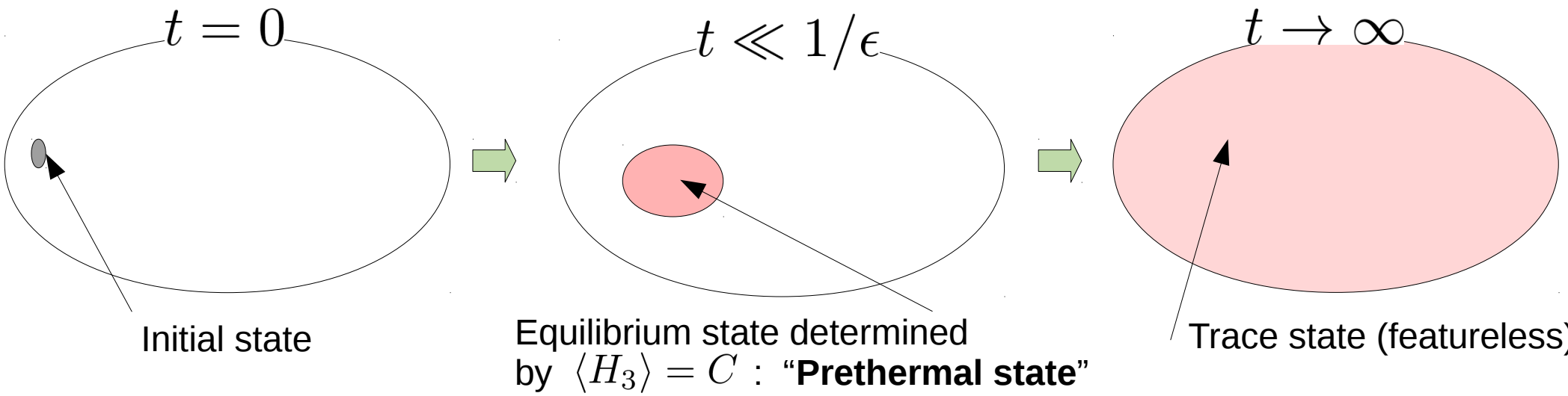
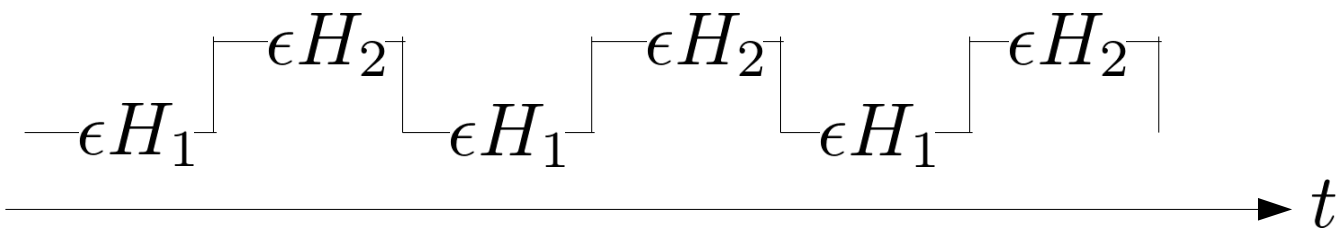


- **Prethermal state**: "Quasi-stationary Noneq state" (Berges, Gasenzer, 2008-...)
- Only the obstruction is sometimes rigorous, not the thermalization and prethermalization (but Kos, Bertini, Prosen 2018)

Simplest example of obstruction: high frequency



$$U = e^{-i\epsilon H_2} e^{-i\epsilon H_1} \stackrel{?}{=} e^{-i\epsilon(H_2+H_1)+i\mathcal{O}_{\rm loc}(\epsilon^2)}$$

Baker-Campbell-Hausdorf?
No, converges only for $||H_i|| \sim 1$

Still, can construct $$H_3 = H_{\rm eff} = \epsilon(H_1 + H_2) + \mathcal{O}_{\rm loc}(\epsilon^2)$$

$$\boxed{U H_{\rm eff} U^* = H_{\rm eff} + \mathcal{O}_{\rm loc}(e^{-1/\epsilon})}$$

$\Rightarrow$ Prethermalization up to exponential times!

(Magnus, …..D'Alesio et al,….. Rigourous 2017: Kuwahara et al, Abanin et al)

Motivation for this work

Replica resummation of the Baker-Campbell-Hausdorff series

(Vajna, Klobas, Prosen, Polkovnikov, PRL 2018)

Kicked many-body model: One-cycle unitary is

$$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}h\sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)}$$

- $|h| + |J| \ll 1$ =====> High-frequency regime

  Exponentially Slow heating & Prethermalization

- $|h| \ll |J| \sim 1$ =====> Moderate frequency but weak driving

  ??????????

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Numerics and Replica Resummation give

$$U^n \approx e^{\mathrm{i}nH_{\mathrm{eff}}}, \qquad n \leq e^{1/\sqrt{\epsilon}}, \quad \epsilon = h \text{ or } J$$

$$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}h\sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)}$$

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Weak driving always gives exponentially slow heating? **No, in general**

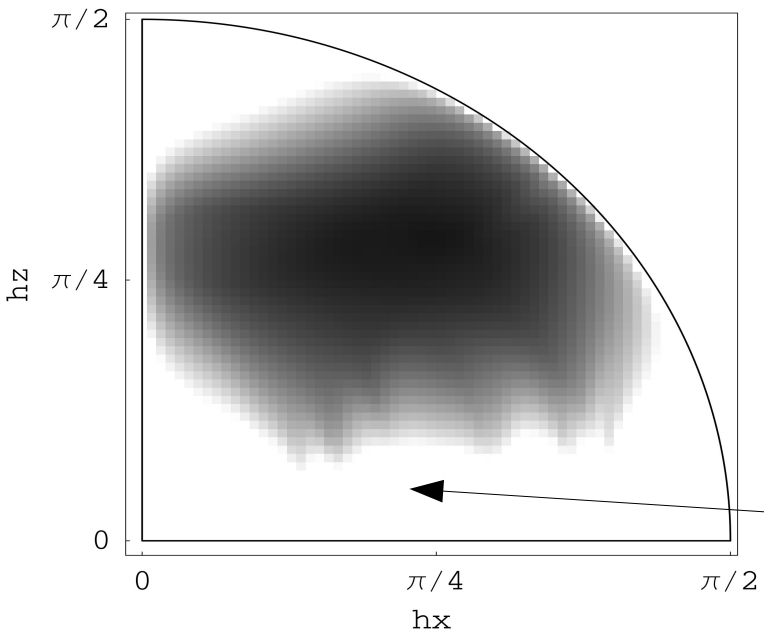
$\mathrm{rate} \sim (\mathrm{driving})^2$

weakly interacting phonons or fermions → kinetic equation

Is there some simple special structure to these models? **Yes: this talk**

A-posteriori motivation

$$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}\sum_i (h_x \sigma_i^x + h_z \sigma_i^z)}$$



Numerics by Prosen 2007:

'Minimal decay rate' Δ of local Ham

$$U H_3 U^* = H_3 + \mathcal{O}_{\mathrm{loc}}(\Delta)$$

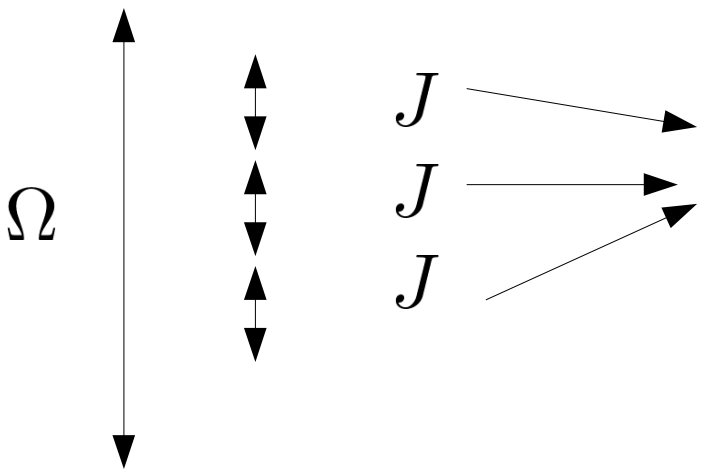
White: $\Delta \leq 10^{-6}$

So it really matters whether h_x or h_z is small

$\Rightarrow$ special structure ?

Recall high-frequency regime

$$H = JH_0 + h\cos(\Omega t)H_1 \qquad\qquad J, h \ll \Omega$$



Many local events needed to absorb one photon of frequency Ω

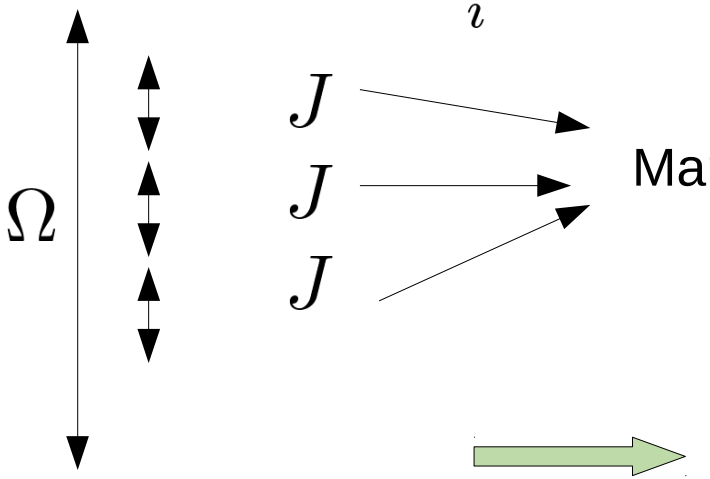
Dissipation only visible in order of PT $\sim \Omega/J$

$$\text{heating rate} \sim e^{-\Omega/J}$$

(Magnus, …..D'Alesio et al,….. Rigourous 2017: Kuwahara et al, Abanin et al)

Same logic: stability of doublons D $(n_i = 2)$

$$H = \Omega \sum_i n_i(n_i - 1) + JH_1 \qquad J \ll \Omega$$



Many local events needed to provide D-energy Ω

$$\mathrm{D-annihilation\ rate} \sim e^{-\Omega/J}$$

(Sensarma et al, ….. Rigorous: Abanin et al, Else et al 2017)

Same logic: stability of doublons D $(n_i = 2)$

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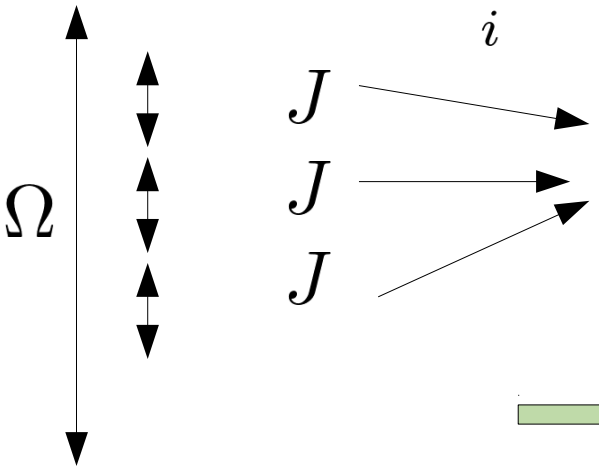


Many local events needed to provide D-energy

Ω

$$\mathrm{D-annihilation\ rate} \sim e^{-\Omega/J}$$

Wait... enough to have two distinct energy scales?

No, crucial propery is: $\Omega \sum_i n_i(n_i - 1)$ can absorb only a discrete small set of energies locally.

Simplest examples:

- Sum of commuting local terms with integer gaps (as here)
- MBL systems (stability of MBL)

So: do we have "sums of commuting local terms with integer gaps" ?

$$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}h\sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)}$$

Yes, both terms have this property =====>

$$\sum_i (\cos(\theta)\sigma_i^z + \sin(\theta)\sigma_i^x)$$

$$\sum_i \sigma_i^x \sigma_{i+1}^x \approx \text{doublons}\, D$$

Choose this one to continue

J

2π (frequency)

To absorb doublon D, need to match frequency up to error of $\mathcal{O}(h)$

n'th order PT: $\min_{m\in\mathbb{Z}} \left|n - m\frac{J}{2\pi}\right| \geq Ch^n$

Mechanism of exp. slow dissipation is there !

Our Theorem

$$H(t) = J(t)D + hW(t), \quad t \in [0,1]$$

Assumptions $x := \frac{1}{2\pi}\int_0^1 dt J(t)$ is "sufficiently Diophantine"

D is sum of commuting local terms with integer gaps

periodicity $H(t) = H(t+1)$

Result Take h small, time $t < e^{C/h^{1/10}}$ and go to rotated frame:

$$i\partial_t \Psi(t) = \widetilde{H}(t)\Psi(t) \qquad [\widetilde{H}(t), D] = 0$$

$\Rightarrow$ D is conserved $\Rightarrow$ no heating

Can expect $\Rightarrow$ **Prethermalization at** $D = \text{constant}$

What means $x := \frac{1}{2\pi}\int_0^1 dt J(t)$ is "sufficiently Diophantine"

Recall: we need n'th order PT: $\min_{m\in\mathbb{Z}} |n - mx| \geq h^n$

Def: x **is Diophantine:** $\min_{m\in\mathbb{Z}} |n - mx| \geq \frac{a(x)}{n^{-\tau}}$ $\tau > 1$

Most numbers are Diophantine:

$$\text{size}\ \{x \in [0,1] : a(x) > \epsilon\} \geq 1 - C\epsilon$$

Our case: $\frac{h}{a(x)}$ **is the real small parameter**

Example and Extension

Recall $$U = e^{\mathrm{i}J\sum_i \sigma_i^x \sigma_{i+1}^x} \; e^{\mathrm{i}\sum_i (h_z \sigma_i^z + h_x \sigma_i^x)}$$

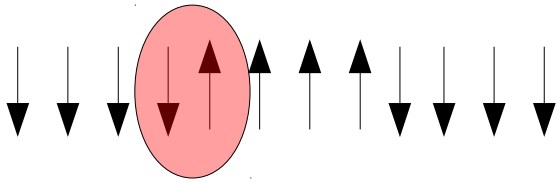
Assume now: $\boxed{h_z \ll h_x \sim J \sim 1}$ instead of $\boxed{h_x^2 + h_z^2 \ll J^2 \sim 1}$

New Diophantine condition: $$\min_{|n_1|+|n_2|+|n_3|\leq n} |n_1 J + n_2 h_x + 2\pi n_3| \geq \frac{a}{n^{-\tau}}$$

Then: Both $$D = \sum_i \sigma_i^x \sigma_{i+1}^x \quad M = \sum_i \sigma_i^x$$ quasi-conserved

kinetically constrained model

Example



No local move possible

Both $D = \sum_i \sigma_i^x \sigma_{i+1}^x \qquad M = \sum_i \sigma_i^x$ quasi-conserved

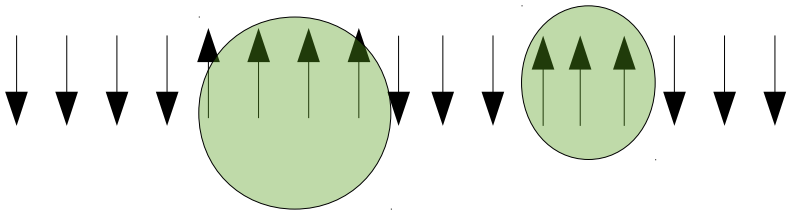
General phenomenology:

- first order in h_z : no spin flips at all
- First dissipation (spin flips) at time h_z^{-4}
- Prethermalization $h_z^{-4} \longleftrightarrow e^{1/h_z^{0.1}}$
- Actually, even at order 4: dynamics is highly constrained

further slowness depending on state

(magnetization, density of doublons)

droplet mass $\sim \exp\left(\text{droplet length}\right)$



- So even prethermalization might be very slow here ------ ‘translation invariant (asymptotic) MBL’

Proof idea: KAM to exhibit conserved quantity

$$H(t) = JD + hW(t), \quad t \in [0,1]$$

- Goal (first order) for some W_d s.t. $[W_d, D] = 0$

$$e^{hA}(-i\partial_t + H)e^{-hA} = -i\partial_t + JD + hW_d + \mathcal{O}_{\mathrm{loc}}(h^2)$$

- Suffices to solve linear ODE with periodic A

$$-i\partial_t A + (W - W_d) + [A, D] = 0$$

- Solution: (write $\mathcal{N} \equiv iJ[D, \cdot]$)

$$A(t) = e^{t\mathcal{N}}A(0) + \int_0^t ds e^{(t-s)\mathcal{N}}(W - W_d)(s)$$

- Imposing periodicity at time t=1 : $A(0) = \boxed{\dfrac{1}{e^{\mathcal{N}} - 1}}(\ldots)$

Resonance Denominator

Conclusion

- Perturbative, rigorous view on slow heating in kicked Ising model

- We identified conditions for slow heating: small perturbations of Hamiltonians with commuting terms + Diophantine

- Not clear whether this indeed explains all the observed absence of heating in this model: numerics needed.