

Contact interactions and Gamma convergence: Bose-Einstein condensate and the Fermi sea..

G.F.Dell'Antonio

Dept. of Math. Sapienza (Roma)

and

Mathematics area SISSA (Trieste)

QUANTISSIMA in SERENISSIMA 2019

SHORT SUMMARY

Using Gamma convergence we give mathematical meaning in Quantum Mechanics to potentials written formally as $\delta(x_i - x_j)$ (strong contact) or $\delta(|x_i - x_j|)$ (weak contact) $x \in R^k$, $k = 1, 2, 3$,

We prove that in a three body system strong contact leads always to an infinite number of bound states, with a scaling law that depends on the system.

We give applications to Low Energy Nuclear Physics, to Bose-Einstein condensation both in the low density and in the high density regimes, to the structure of the Fermi sea in Solid State Physics and to the structure of the bound states in the Nelson model (interaction of a particle with a zero-mass quantized field).

This talk is about using Gamma convergence in Quantum Mechanics as a tool to construct self-adjoint extensions in the case of contact interactions.

Gamma convergence is a variational technique of common use in the theory of composite and very fragmented materials [Dal]. It was introduced by R. Buttazzo and E. de Giorgi over sixty years ago and is of common use in Applied Mathematics.

It is much related to *homogenization*, an approach in which to describe composite fragmented materials one takes first a *magnifying glass* to study the details of the structure and then draws conclusions about the macroscopic properties.

With this approach we study "contact interactions", self-adjoint extensions constructed with "potentials" that are distributions supported by the lower dimensional manifold $\{x_i = x_j\} \ i \neq j$ and are invariant under rotation.

Our strategy is to follow the approach of the classical case, regarding the perturbation as quadratic form. We will see that contact interaction are associated *formally* to delta potentials. To have self-adjoint operators, the first step is to introduce a map to a space of more singular functions. By duality the potential is more regular. This map is mixing and fractioning in a precise way.

Our approach is somewhat related to the approach of Birman, Visik and Krein [B][K][V] *but on the side of quadratic forms* [A,S][K,S]. Therefore we call the map *Krein map* \mathcal{K} . The idea of the strategy we adopt came from reading [M2]. Therefore we call *Minlos space* \mathcal{M} the target space.

The map acts differently on the free hamiltonian and on the potential part.

This reflects the fact that the free hamiltonian is an operator while the potential can only be seen as a quadratic form which is not strongly closed.

Therefore going back to "physical space" *is not inversion of the map.*

The Krein map is mixing (it is not diagonal in the channels) and fragmenting (the target space is made is made of more singular functions).

Under the map, the hamiltonian is mapped into a continuous family of self-adjoint operators (the operator has in \mathcal{M} a quasi-homogenous stricture) [D,R].

If the interaction is strong enough, each of them has an infinite number of eigenvalues that diverge linearly to $-\infty$.

Going back to physical space these operators are turned into quadratic forms bounded below *but only weakly closed*.

Gamma convergence select one (the infimum) that can be closed strongly [K]. This is the hamiltonian of our system.

Gamma convergence implies strong resolvent convergence. [Dal]

We will exemplify this method in Low Energy Nuclear Physics, in Bose-Einstein condensation (both in the low and in the high density regime), in the description of the Fermi sea in Solid State Physics for particles that satisfy the Pauli equation and in the construction of the ground state in the Nelson model (interaction of a particle with a quantized zero mass field).

Our analysis can be applied to other cases where there is an Efimov sequence of bound states, e.g. in a three particle system with two zero energy resonances $[S][T][O,S]$ and in the case of a quantum particle in a potential which has the form of two zero energy resonances $[A,S]$.

Since the method uses Sobolev semi-norms the procedure does not apply in general to interactions in Fock space; still we construct a baby field theory model with particle-antiparticle creation.

Remark that we will mainly consider wave functions and their structure. To make contact with existing literature on Bose-Einstein condensate *which deals with densities* (positive trace class operators) one should consider the weak form of the equations .

If the particles are identical, taking the scalar product with the wave function of a particle and integrating by parts the kinetic term, we obtain a functional that has a term is quadratic *in the density*. The variational equation for this functional has an interaction part that cubic attractive with coefficient the Gross-Pitayewski constant in the weak contact case and a different coefficient in the high density case.

The solution is now seen as critical point of a functional which is the sum of a kinetic term and a local term which is quadratic *in the densities*. The ground state of the system is in both cases the tensor product of three wave functions *but this is not visible in the formalism of density matrices*.

It is worth remarking that while on some of the existing mathematical literature [B,O,S] on Bose-Einstein condensation the resulting equations are the result of interactions with range that depends on the number of particles, we obtain the Gross-Pitayeskii and the cubic Schrödinger equation as a result of zero-range interactions; also here in the Gross-Pitayewkii case there must be a zero energy resonance.

We will prove that the zero range interactions are limits, *in the strong resolvent sense*, of potentials that scale as $V^\epsilon(y) = \frac{1}{\epsilon^\alpha} V(\frac{y}{\epsilon})$ where $\alpha = 3$ for strong contact and $\alpha = 2$ for weak contact.

Notice that in our formulation the parameter is the range of the interaction and not the number of particles as in [B,O,S].

CONTACT INTERACTIONS

In Quantum Mechanics contact (zero range) interactions in R^3 are self-adjoint extensions of the symmetric operator as the free hamiltonian restricted to functions that vanish on the *contact manifold* Γ

$$\Gamma \equiv \cup_{i,j} \Gamma_{i,j} \quad \Gamma_{i,j} \equiv \{x_i - x_j\} = 0, \quad i \neq j \quad x_i \in R^3 \quad (1)$$

On Γ it required that the function in the domain of these extension satisfy the boundary conditions $\phi(X) = \frac{C_{i,j}}{|x_i - x_j|} + D_{i,j} \quad i \neq j$

These conditions were introduced already in 1935 by H.Bethe and R.Peirels [B,P] in the description of the interaction between proton and neutron.

The problem of zero range interaction was first analyzed from a mathematical point of view by B.S.Pavlov [Pa] who investigated in the weak contact case, self-adjoint extensions defined by the condition that the wave function takes a finite value at the boundary

A further analysis was done by Yu Shondin [Sh] in the case of separate weak contact, following a scheme for self-adjoint extension led out by Yu Shirokov.

Later the problem was analyzed by R.Makarov [Ma] R.Makarov and V.Melezdik [M,M].

In 1962 Minlos and Faddeev [M,F] proved that joint strong contact of three particles leads to a hamiltonian which is unbounded below.

Minlos [M1][M2] studied the case of strong separate contact in a three particle system, concentrating on (the physically relevant) case of two identical particles interacting separately through "zero range potentials" with a particle of the same mass. We concentrate on this case.

Remark

Here we follow the misleading tradition to use the name "particle" for a wave function. In Quantum Mechanics a particle is a density matrix (a probability distribution).

We shall come back later to this point, that enters crucially e.g. in the description of a Bose-Einstein condensate.

Contact interactions were used by Skorniakov and Ter-Martirosian [S,T] in their analysis of three body scattering within the Faddeev formalism. We shall call them Ter-Martirosian [T-M] boundary conditions.

At the boundary the functions we consider are not in the domain of the free hamiltonian; *solution* of the Schrödinger equation is only meant in a weak sense.

In the Heisenberg representation the T-M boundary conditions are described *FORMALLY* by potentials $V_{i,j}(|x_i - x_j|)$ and $U_{i,j}(|x_i - x_j|)$ that are *distributions supported by the boundary* Γ (the support is the same but the strength is different).

$$V_{i,j} = -C_{i,j}\delta(x_i - x_j) ; U_{i,j}(\rho_{i,j}) = -D_{i,j}\delta(|x_i - x_j|) \quad C_{i,j} > 0 \quad D_{i,j} > 0 \quad (2)$$

This can be verified by taking the scalar product with a function in the domain of \hat{H}_0 (the free hamiltonian restricted to functions that vanish in a neighborhood of Γ) and integrating by parts twice.

We call *strong contacts* the self-adjoint extension characterized by the constants $D_{i,j}$ and *weak contacts* the one characterized by $C_{i,j}$.

Notice that the support of these two interactions is the same but their "strength" is different.

Since both are rotation invariant they affect only s-waves.

Weak contact defines a self-adjoint operator that has in the weak closure of its domain a zero energy resonance (a solution of $H\psi = 0$ that behaves as $\frac{1}{|x_i - x_j|}$ at infinity).

FORMALLY this is seen by the identity

$$\Delta \frac{1}{|y|} = -C(|y|\delta(y))\left(\frac{1}{|y|}\right) \quad (3)$$

The function $\frac{1}{|x|}$ is in the weak closure of $L^2(\mathbb{R}^3)$. This implies a topological property (the lack of compactness of the domain in a Sobolev topology).

We will give later a precise formulation.

This implies that if a sequence of potential V^ϵ is such that $H_0 + V^\epsilon$ converges in strong resolvent sense to a weak contact hamiltonian the hamiltonian $H_0 + V^\epsilon$ must have a zero energy resonance.

Point interaction [A] is weak contact with an infinitely massive particle(fixed point).

For two bodies the weak contact potential is defined on a core of the quadratic form of the laplacian.

Therefore weak contact can be described as a form perturbation of the laplacian.

In three dimension strong contact between two bodies is not defined.

On the contrary in a three particle system *separate* strong contact of one particle with two particles can be defined and is represented by a self-adjoint operator (with Efimov spectrum) .

The only function of the third particle is to add extra degrees of freedom.

Remark again that the use of the name "particles" is somewhat inappropriate; the name "wave function" is more correct.

If the two particles are identical the strong contact *is with a density*. We will come back to this point later.

Weak contact hamiltonians are limit in the strong resolvent sense, of hamiltonians with potentials that scale as $V^\epsilon(|y|) = \frac{1}{\epsilon^2}V(\frac{|y|}{\epsilon})$ and have a zero energy resonance.

This result, well known for point interaction, follows easily from our analysis of strong contact.

For weak contact our analysis simplifies the proofs since it permits to treat separately the singularity on the resolvent due to the resonance and that due to the singularity of the interaction (one in a short distance problem, the other a long distance one).

We concentrate first on strong contact . Part of the problem is to give mathematical substance to these hamiltonians.

For this purpose we introduce a map \mathcal{K} which can be considered as a magnifying glass. It is "mixing and fragmenting" in well defined sense.

The target space contains more singular functions. By duality the operator has a more regular structure.

For the purpose of constructing the Krein map we use the assumption that the system contains a third non-interaction particle. We choose this map to be the action of $H_0^{-\frac{1}{2}}$ where H_0 is the free hamiltonian of the three-body system.

Under this map the free hamiltonian and the potential transform differently: the kinetic energy as an operator, the potential as a quadratic form (notice that such potential can only be defined as quadratic form).

in the target space the free hamiltonian acts as $\sqrt{H_0}$.

In the target space the potential is the convolution of the delta potential with the resolvent $\frac{1}{H_0}$ (recall that the delta function commutes with H_0).

Notice that the mapping operator is not diagonal in the particle variables.

This map is "fractioning" (the image space is more singular) and "mixing" (the map is not diagonal in the coordinates of the two particles).

To see the connection with the procedure used by Krein notice that the delta potential commutes with the free hamiltonian so that the map on the potential can be written $\delta \rightarrow \delta H_0^{-1}$.

The result for the potential is the sum of $\frac{-C}{|x|}$ and a positive bounded operator. The positive constant C depends on the mass of the particles and is proportional to the coefficient of the delta. Therefore in \mathcal{M} the hamiltonian is quasi homogeneous $[D,R],[L,O,R]$ of order one.

It follows that there exist C_1 and $C_2 > C_1$ such that in \mathcal{M} in the zero angular momentum sector if $C \geq C_1$ the hamiltonian has a Weyl limit circle degeneracy and if $C \geq C_2$ is unbounded below.

This results when $C \geq C_1$ in a one-parameter family of self-adjoint hamiltonians; if $C \geq C_2$ each hamiltonian has a sequence of bound states that diverge linearly to $-\infty$.

The wave functions of the bound states behave at the origin in polar coordinates as $\frac{c}{|\rho \log^2(\rho)|}$. This result is obtained in [M2] solving the equation $H_0 = H_0^*$.

Due to the change in metric topology, inverting the Krein map one has a sequence of quadratic forms which are uniformly bounded below *but only weakly closed* .

Gamma convergence [Dal] SELECTS ONE (the infimum); its strong closure is the hamiltonian of strong contact. If the interaction is strong enough this self-adjoint extension has an Efimov sequence [E] of bound states.

Gamma convergence implies strong resolvent convergence [Dal] (but not quadratic form convergence) and therefore convergence of the Wave operator in Scattering Theory and of spectra for self-adjoint operators.

We will see that strong contact interaction has a role also in Bose-Einstein condensation in the high density regime and also in the description of the Fermi sea in Solid State Physics.

One may wonder what is the role of the extra family of weakly closed quadratic forms. They correspond to critical point of the action functional on quantum states that have the "wrong" boundary conditions at Γ (e.g. for the presence of a strong magnetic field).

If one defines the laplacian in H_0 using some other boundary conditions the operator chosen by Gamma convergence *changes accordingly*.

Since the construction relies on a variational argument *no rate of convergence is obtained*. The method *is not perturbative*..

In Low Energy Physics the first few elements of an Efimov sequence [E] of bound states have been found experimentally. It is easy to verify that these Efimov states (called "trimers" in the Physics Literature) have increasingly larger essential support.

This self-adjoint extension has a sequence of bound states with energies that scale geometrically.

Consider now a sequence of two-body hamiltonians with the the same free part and a sequence of potential s $V^\epsilon(x) = \frac{1}{\epsilon^3}V(\frac{|x_i-x_j|}{\epsilon})$ where the L^1 norm of the negative potentials is equal to the coefficient of the delta potential.

Their quadratic forms are a strictly decreasing sequence; since there are no zero energy resonances they belong to a subset that is compact for the topology given by Sobolev semi-norms.

Since they are uniformly bounded below and the set is compact (there are no zero energy resonances) there is a minimizing sequence.

Since the forms are bounded above by the form of strong contact the limit form coincides with the unique limit provided by Gamma convergence.

Gamma convergence implies strong resolvent convergence [Dal] (*but not quadratic form convergence*). The hamiltonian we have found is therefore the limit, in strong resolvent sense, of hamiltonians with potentials that scale (in R^3) as indicated before.

This result is the main justification of the method we use. *No rate of convergence can be given: the result is non perturbative.*

The Gamma limit of a sequence of *strictly convex* weakly closed forms F_n in a topological space Y is the *unique* weakly closed quadratic form F such that for any subsequence the following holds

$$\begin{aligned} \forall y \in Y, \quad y_n \rightarrow y, \quad F(y) &= \liminf_{n \rightarrow \infty} F(y_n) \\ \forall x \in Y \exists \{x_n\} \rightarrow x \quad F(x) &\geq \limsup_{n \rightarrow \infty} F(x_n) \end{aligned} \quad (4)$$

The first condition means that F provides an asymptotic common lower bound for the F_n ; the second condition says that this lower bound is optimal.

The limit form admits strong closure

The condition for the existence of the Gamma limit is that the sequence be contained in a compact set of the topological space Y . In the present setting Y has the Frechet topology given by Sobolev seminorms

Compactness of bounded set is in our case (strong contact) a consequence of the absence of zero energy resonances. Therefore there exists a minimizing (Palais-Smale) sequence.

We will consider later the case of "weak contact" in which the contact hamiltonian has a zero energy resonance so that our previous arguments fails.

In this case we approximate the weak contact hamiltonian with hamiltonians with potentials that lead to a zero energy resonance.

In this case the difference of the quadratic forms of the weak contact hamiltonian and of the approximation hamiltonians $V^\epsilon(x) = \frac{1}{\epsilon^2}V(\frac{x}{\epsilon})$ is in a compact domain and Gamma convergence takes place.

We will study later the case of three particles which are pairwise in weak contact (this will enter in the analysis of the dilute Bose-Einstein condensate) .

To prove resolvent convergence of the approximating hamiltonians in this case we have to rely on the result of the case of strong contact.

We prove that strong contact hamiltonians are limits, in the strong resolvent sense, when $\epsilon \rightarrow 0$ of hamiltonians with two-body potentials $V^\epsilon(x_i - x_j)$ with $V \in C^1$ that scale as $V^\epsilon = \frac{1}{\epsilon^3} V\left(\frac{|x_i - x_j|}{\epsilon}\right)$.

This follows because the sequence of two-body hamiltonians with potential V^ϵ converges from above (the potential is negative, its L^1 norm increases without bound) and the (unique) Gamma limit is a lower bound). No rate of convergence can be obtained.

In the weak coupling case is easy to see that a zero energy resonance is in the weak closure of the domain.

Therefore compactness fails.

Weak two-body contacts are limit of hamiltonians with two-body potentials $V \in C^1$ that scale as $V^\epsilon = \frac{1}{\epsilon^2} V(\frac{|x_i - x_j|}{\epsilon})$; this can also be proved with standard methods of Functional Analysis.

This can be cured by performing a map to a weaker Sobolev space. In this space the "resonance" is an element of the space, and compactness is restored.

The proof using the Krein map has the advantage that the map separates in the resolvent the singularities due to the zero energy resonance from those due to the zero range.

The hamiltonians with potentials that scale as $V^\epsilon(|y|) = \frac{1}{\epsilon^2} V(\frac{|y|}{\epsilon})$ converge in strong resolvent sense to the hamiltonian of weak two-body contact.

Inverting the map one concludes that In the original space weak convergence fails, but the difference of the resolvent for the approximate potential and the resolvent of weak contact converges strongly.

In the same way one proves that converges strongly to zero the difference of the resolvent of separate weak contact of one particle with two particles with the resolvent of a system of three particles of which one interact separately with the other two through a potential that scales as V^ϵ .

Moreover one proves that the limit hamiltonian has a bound state and no zero energy resonances (the resolvent at zero energy is an invertible two-by-two matrix with zeroes on the diagonal):

Things are more interesting in the case of *simultaneous* weak contact of three particles.

In the Birman-Schwinger formula for the resolvent each potential provides a $\frac{1}{\epsilon^2}$ factor.

The terms in the perturbation series that have a contribution from all three potentials have a factor $\frac{1}{\epsilon^6}$ that can be attributed equally to two of the potentials.

The result is the perturbation series for separate interaction of one particle with the other two with a potential that scales with a $\frac{1}{\epsilon^3}$ factor.

We have proved that this leads to a perturbation series that converge, when $\epsilon \rightarrow 0$ to the resolvent of the hamiltonian of strong separate contact of a particle with two particles.

Therefore the limit hamiltonian has an Efimov sequence of bound states.

The advantage in using the Krein space is that the singularities in the resolvent are clearly separated, and this leads to a simplification in the proofs as compared to [A] (point interaction is weak contact with an infinitely heavy particle)

LOW DENSITY BOSE-EINSTEIN CONDENSATE

The analysis can be adapted to study the Bose-Einstein condensate both in the low and in the high density regime.

We make use of the results in the strong contact case.

In the usual formulation of the theory of Bose-Einstein condensation the Bose gas is confined by a strong potential and the mean distance between bosons is assumed to be of order ϵ where ϵ is a small parameter.

In the low density regime we can that the interaction of the particles in the gas is a surface effect and the particles feel an attractive force with potential $V_\epsilon(x_i - x_j) = \frac{1}{\epsilon^2} V\left(\frac{x_i - x_j}{\epsilon}\right)$ (weak coupling).

The weak coupling is produced by a zero energy (Feshbach) resonance.

Recall now that a particle is represented by a density matrix; in a wave function formulation the interaction is therefore with two identical "particles". It is unfortunate that usually one uses the name "particle" to denote a wave function. In the same spirit it is unfortunate that one refers as "bound state" to an eigenstates of the Schrödinger operator (that may be complex-valued) and its modulus square).

We regard the Bose-Einstein gas as a collection of particle pairs in strong contact (high density case) or in weak contact (low density case); their interaction is to a large extent independent of the presence of other particles (one can show, using the Konno-Kuroda [K,K] formulation of the Birman-Schwinger resolvent equation, that strong contact, weak contact and regular perturbations lead to complementary and independent effects).

Weak and strong contact may originate from densities that force two particles to be very close, but they may have a different origin as is the the case in low energy nuclear physics.

In that case they lead to the existence of Efimov states which are bound states of three particles (Trimers),

The presence of an Efimov series of trimers (recognized as such by the scaling of energy of the bound states) has be detected in experiments.

Notice that we refer to a Bose-Einstein condensate, both in the low density and in the high density regime, as a condensate *bound states of pairs of particles*.

Indeed, since a particle is represented by a density matrix an interaction term which connects three wave functions of identical particles corresponds in the density matrix formulation to a two particle interaction (in the density matrix formulation to obtain an energy functional one has to take the scalar product with a wave function and integrate by part the kinetic energy term).

Since weak coupling is the limit of interaction through a potential that scale as ϵ^{-2} the ϵ^{-1} missing (as compared to strong interaction) can be attributed to a low density [B,O,S]

To make contact with [B,O,S] notice that the extra factor L^{-1} that appears there corresponds here to the choice of weak contact.

Due to the presence of *two* zero energy resonances there is a bound state; we denote by Ω_w the wave function. This bound state is stable because the hamiltonian of the two particle subsystem is positive (and has a zero energy resonance).

The particles are identical and satisfy Bose-Einstein statistics.

The ground state of the system of $2N$ particles is (approximately) the tensor product of the vectors $\otimes \Omega_w^i$ for all different two-body pairs (properly symmetrized since the particles are identical bosons). The error is of order $\frac{1}{N}$.

We emphasize that the low density Bose-Einstein condensate is a condensate of *weakly bound particle pairs*.

Choosing $\epsilon \equiv \frac{1}{N}$ permits a Fock space analysis. We will not analyze further here this problem .

BOSE-EINSTEIN CONDENSATE, HIGH DENSITY CASE. THE NEW GROUND STATE

Consider now the high density case. The interaction is represented, before taking the limit $\epsilon \rightarrow 0$ by the hamiltonian

$$H_{int} = H_0 + \sum_{i \neq j \neq k} \frac{1}{\epsilon^3} V\left(\frac{|x_i - x_j|}{\epsilon}\right) \quad (5)$$

We have remarked that one can alternatively consider a weak contact among all three wave functions).

In the perturbation formula for the resolvent the terms that depend only on two of the potentials give the same result of the weak contact interaction of one "particle" with a pair. We are interested in the contribution of terms *that depend on all three potentials*.

In this contribution we can "artificially" take away two ϵ from the denominator of one of the potential and "give" an ϵ^{-1} factor to each of the other two (this artifice does not alter the result).

The remaining potential now plays no role.

Redistributing the ϵ is an artifice but it leads to the conclusion that at high density *the ground state of system* is better described considering a system of three particles one of which is in *strong* contact with the other two.

The strong contact interaction takes place *separately* with the two "particles" (which do not interact; since the particles are identical one has a gas of two particles in strong contact).

Remark that the presence of a third "particle" is mandatory to define strong contact. The role of the third particle is to prevent free motion for the barycenter of the two particles in strong contact.

Call Ω_s the ground state. To first order the ground state of the high density Bose-Einstein gas is $\otimes_i \Omega_s^i$.

It is not related to the ground state $\otimes_i \Omega_w^i$ of the diluted Bose-Einstein gas.

Since the two (identical) bosons in the pair are in strong contact, each of them satisfies the Schrödinger equation with as potential the density of the other i.e. the focusing cubic Schrödinger equation (and not the Gross-Pitaewskii equation which has *a different effective coupling constant* due to the presence of a zero energy resonance) [E,S,Y].

Since the system has two zero energy resonances the inverse of the resolvent is at the origin a two-by-two matrix with zeroes on the diagonal. It has therefore two eigenvalues of opposite sign and therefore there is a bound state.

This produces a three "particles" (wave functions) bound state.

The ground state of the Bose-Einstein gas is the symmetrized tensor product of these three "particle" states. Since the particles are identical it is actually a tensor product of two-particles wave functions..a wave function with a density matrix.

Since there is a resonance the system satisfies the Gross-Pitayewski equation.

If all three particles have a weak contact in the Birman-Schwinger formula there is a contribution coming from all three weak contact potentials. This may be seen as high density regime but the role of density is not clear.

This contribution has the same power of ϵ as the strong separate contact of one of the particles with the pair (one can distribute the factor ϵ^{-2} evenly on two of the potentials).

If the two particles are identical *it is as if* the coupling be locally to the nucleus of the density matrix $|\psi(x)|^2$.

The ground state of the system is again a tensor product (but of different states).

If the system is described by density matrices one should consider a weak form of the equation,.

In the limit $\epsilon \rightarrow 0$ the other particles do not part in the interaction. If one relates the strength of the interaction to the density of the gas, this is a low density setting.

But the interaction between a pair is very strong.

Taking the scalar product of the wave function of the system with the wave function of one particle and integrating by parts the kinetic term the solution is seen as critical point of a quantum functional that is the sum of a kinetic term and a local functional that is quadratic in the density.

The negative coefficient is the Gross-Pitayewskii coupling constant in the weak coupling case and the and the coefficient of the cubic term in the strong contact case.

It is interesting to compare our approach to Bose-Einstein to that of [B,O,S]

In our approach the small parameter ϵ is a measure of the range of the potential.

The result is independent of the number of particles but the number of particles enters if we refer to a condensate which is confined in a cubic box of side 1 and we require that the box contains N particles. *dilute* refers to the case $\epsilon \simeq \frac{1}{N}$ and *dense* to the case $\epsilon \simeq \frac{1}{N^3}$ (close packing).

In our terminology this is respectively weak and strong contact.

In [B,O,S] the parameters are chosen such as to have an approximation to weak contact, but with a different wording and not as self-adjoint operator.

We remark that in [B,O,S] the fact that weak contact is a two-body problem is not emphasized and the more difficult problem of strong contact is not considered.

We have so far taken the point of view of self-adjoint extensions.

The point of view which considers energy functionals [L] is very different.

The quantum particle is represented by a probability distribution.

The potential couples two "physical" particles.

Since the coupling is through a strong interaction the energy functional is

$$E = \int (\nabla\phi, E_0\nabla\phi)d^3 - g \int \phi(x)^2\phi(x)^2dx \quad (6)$$

Using Morse theory and variational analysis one sees that there are constants g_1 and g_2 such that if $g < g_1$ the functional has no critical points, if $g_1 \leq g < g_2$ there is a critical point and if $g \geq g_2$ there are infinitely many points that scale as $\frac{1}{n}$, No need of self-adjoint extension!

The case of strong contact was considered by Lieb et al.. from a variational point of view. The (approximate) ground state was found, but the existence of an infinite set of bound states was not clarified.

Gamma convergence is a tool that can be used effectively in operator theory through the use of the Krein map.

REMARK

It is interesting that the same structure (Efimov effect and strong contact) occurs in Low Energy Physics and in the Bose Einstein gas: in the former it is due to the interaction within nuclei, in the latter is due to pressure of the gas.

We shall see that the same phenomenon occurs in Solid State Physics; this time it is due to action of the nearby nuclei on the conduction electrons.

SEMICLASSICS

It is interesting to notice that substituting H_0 with $H_0 + \lambda$, $\lambda > 0$ one has a family of maps.

$$\sqrt{H_0 + \lambda} = \sqrt{\lambda} + \frac{1}{2} \frac{H_0}{\sqrt{\lambda}} + O(\lambda^{-\frac{3}{2}}) \quad (7)$$

Setting $\frac{1}{\sqrt{\lambda}} = \hbar$, apart from a constant term the kinetic energy in Krein space is to first order in \hbar the free hamiltonian of the quantum system.

In the Minlos space \mathcal{M}_λ strong contact potentials are represented by the Coulomb potential $-\frac{C}{|x_i - x_j|}$ $C > 0$.

Therefore for λ large \mathcal{M}_λ can also be regarded as semiclassical space.

In the semiclassical limit the free hamiltonian is scaled by a factor to \hbar^{-2} and the Coulomb potential is scaled by a factor \hbar^{-1} .

If we identify the radius of the potential (the parameter ϵ) with \hbar (both have the dimension of a length) the limit $\hbar = \epsilon \rightarrow 0$ gives contact interaction at a quantum scale, Coulomb interaction at a semiclassical scale.

Therefore the Newtonian three body problem can be considered as semiclassical limit of the quantum three body problem of a three particle system in which each particle is in weak contact with the other two.

In the semiclassical limit the quantum mechanical energy functional reduces to the classical energy functional for the three-body Newton problem, the quantum particles are represented by coherent states and the (infinitely many) bound states are now the infinitely many periodic solutions of the classic newtonian three-body problem

Addition of a magnetic potential is represented as usual with the substitution $i\nabla \rightarrow i\nabla + A$.

If the magnetic term of the hamiltonian is singular at the boundary $\Gamma_{i,j}$ the hamiltonian selected by Gamma convergence is the one that takes into account the presence of a singular magnetic field (new boundary conditions).

THE TWO-DIMENSIONAL CASE

The analysis we have done for $d=3$ can be repeated in the case of two space dimensions. Also in this case one can define weak and strong interactions through the use of a Krein map (defined as in $d=3$).

For $d = 2$ in a two particle system one can have both a strong and a weak contact interaction.

In a three-particle system strong contact among all three particles leads to a system that has an Efimov sequence of bound states and to Bose-Einstein condensation.

We do not give here the details.

THE FERMI SEA

Consider now the motion of the conduction electrons in a crystal.

Conduction electrons are spin $\frac{1}{2}$ particles that satisfy the Pauli equation, a first order equation with hamiltonian $H_P = i\sigma \cdot \nabla$ where σ_k , $k = 1, 2, 3$ are the Pauli matrices.

Their motion is constrained by the joint action of the three neighboring nuclei.

As a result the motion is restricted to a small neighborhood of a Y shaped graph as suggested also by pictures taken with an electron microscope [Am].

In a neighborhood of the vertex the restriction is due to the combined action of the three nuclei, in a neighborhood of the edges the restriction is due to two atoms. Estimates are in [W, T].

The interaction is attractive (from the point of view of the nucleus we consider) because after the interaction the conduction electron of a nucleus is forced to move closer.

Remark that Solid State Physics is now considered a field of research in which dynamics takes place on a lattice.

While this approach has produced remarkable results, it is interesting to find a connection with an underlying more fundamental quantum mechanical structure.

In particular this may lead to clarify the notion of "Fermi sea" which plays a basic role in the lattice theory but is difficult to understand from the point of view of Quantum Mechanics.

The interaction takes place at the vertex.

Since the Pauli equation is linear, we can take as variables the difference of the coordinates (with the origin at the vertex of the graph). Notice that this is possible for fermions because the particles have spin $\frac{1}{2}$.

This choice of coordinates makes the problem analogous to the joint contact in dimension two of three particles that satisfy the Schrödinger equation.

There are two main differences: the dimension is now one and the particles are spin $\frac{1}{2}$ fermions that satisfy the Pauli equation and Fermi-Dirac statistics..

Since the direction of momentum changes at a vertex and the equation is of first order It is natural to describe the interaction potential by a one-dimensional delta function.

Denote by $H_S = \sqrt{H_P \cdot H_P^*}$ the Salpeter hamiltonian and by H_S^3 the Salpeter hamiltionian for the three body problem.

Since the Pauli matrices anticommute, $H_S^3 = \sqrt{H_{P,3} H_{P,3}^*}$ where $H_{P,3}$ is the Pauli hamiltonian for the three body problem.

The Krein map is now induced by the operator $\sqrt{H_S^3}$. This map is again mixing and fractioning.

For our purposes is more convenient to use as hamiltonian *the positive part* of the Pauli operator, i.e. the Salpeter hamiltonian.

This is not strictly necessary (for the kinetic term the inversion of the Krein map coincides with returning to the original operator) but it avoids having two negative operators.

Under the Krein map the kinetic energy is a positive differential operator of order $\frac{1}{2}$.

There are three simultaneous interactions with singularity of order one (the three deltas).

As in the case of three weak contact incase of Schröduger equation it follows from the Birman-Schwinger formula that the result is the same as if one considers that the system is composed of a particle interacting separately with a pair of particles through a two body potential that has a singularity order $\frac{3}{2}$. This potential is an artifact. Its only purpose is the prove resolvent convergence of the hamiltonian with the potentials V^ϵ .

Recall that an artifact is also the delta in the three-dimensional case. I

The Krein map provides the convolution of the resolvent of the Salpeter hamiltonian with the potential ; this produces a term which has a singularity of order $\frac{1}{2}$ in position space with a negative constant $-C$ that depends on the strength of the interaction.

the residual terms give weak contact and regular interaction that do not affect the result.

In \mathcal{M} the system is "almost homogenous" : the degree of the differential operator is $\frac{1}{2}$ and it is equal to the degree of singularity of the potential at the origin.

It follows $[D,R][L,O,R]$ that there are constants $C_1 < C_2$ such that for $C \geq C_1$ there is a one-parameter family of self-adjoint operators.

For $C \geq C_2$ each member of the family has an infinite number of negative bound states that are asymptotically proportional to $-\sqrt{n}$.

Inverting the Krein map one has an ordered family of weakly closed forms bounded below. Gamma convergence selects the infimum.

This form can be closed and defines a self-adjoint operator bounded below with an Efimov spectrum.

The eigenvectors have at the origin the asymptotic behavior $\frac{c_n}{|x|^{\frac{1}{2}} \log^2(n)}$ and the n^{th} eigenvalues scales as $\frac{1}{\log n}$.

The bound states are therefore very extended.

Gamma convergence implies strong resolvent convergence when $\epsilon \rightarrow 0$.

In an extended crystal we use periodic boundary conditions.

Recall that electrons are identical spin $\frac{1}{2}$ particles and satisfy the Pauli exclusion principle (no more than two electrons can be in the same bound state).

Therefore if there are $2n$ electrons they occupy the lowest n states.

But there are infinitely many bound states (a Hilbert hotel) and therefore however large the crystal is all electrons are in a bound state (*Fermi sea*).

Since their wave functions decay in absolute value in each direction as $\frac{1}{|x|^{\frac{1}{2}} \log^2(n|x|)}$ for large crystals most of the bound states have very flat wave function and the energies of these bound state converge to zero with a $\frac{1}{\log n}$ law.

Most of the bound states *are near the surface of the sea*

The Fermi sea, *a physical object*, should not be confused with the *Dirac sea*, an artifact used in a relativistic setting.

For a large enough crystal the Fermi sea (or Fermi band, i.e. an interval of energy with macroscopic population) can be very densely populated and can appear as a continuum .

If the spectrum of the "Fermi sea" is added to the spectrum of the crystal of the nuclei *the spectral gaps may be closed*.

In a sufficiently large crystal the electrons at the surface of the Fermi sea have practically no binding energy.

This justifies the "Dirac-like" behavior of their wave functions and implies that a current is produced by a very small electric field (superconductivity).

The decay of the eigenvalues is only logarithmic. The wave functions are very extended and may look locally as plane waves..

If there is a relevant amount of impurities (or if the crystal is random) more levels are added, therefore there are more states that must be occupied.

If the density of impurities is sufficiently high (or if the crystal is "sufficiently random") the last filled level has a finite negative energy (at a semi-classical level this is described by positive potentials that "slow down" the diffusion).

Therefore a small electric field is not sufficient to "extract an electron " and the conductivity of the sample decreases sharply.

In a sufficiently large crystal the wave function of the electrons *at the surface of the Fermi sea* is very extended and there is practically no binding energy.

This justifies the "Dirac-like" behavior of their spectra and implies that a current is produced by a very small electric field (superconductivity).

Since the decay of the eigenvalues is only logarithmic, for a large enough crystal the Fermi sea (or Fermi band, i.e. an interval of energy with macroscopic population) can be very densely populated and appear as a continuum. This is perhaps the origin of the notation "half filling".

If this atomic spectrum is added to the spectrum of the crystal of the nuclei *the gaps may be closed*.

Notice that "most" of the wave functions of the electrons are very extended.

This implies that in three dimension in a large crystal the wave function of most of the conduction electrons has essential support at a distance $O(N^{\frac{1}{3}})$ from the border of the crystal.

Conduction is a "border effect" .

In presence of many local perturbations the number of bound states increases sharply (each configuration has its wave functions).

The highest occupied state has a non zero binding energy and conductivity drops.

There is a further interaction of the electron: it is the interaction that takes place along the edges and is due to the combined action of two nuclei.

This interaction is much weaker and can be described as non-singular interaction of an electron along the edge.

One can prove that weak and strong contact have independent and complementary effects (and independent and complementary effects from those due to smooth interactions).

Therefore the main features are those of strong contact.

For the proof one relies on a Konno-Kuroda formulation $[K, K]$ that allows the separation of the different scales.

Electrons at the two ends of an edge of the graph form a magnetic dipole; in presence of a magnetic field the orientation of the dipole is changed.

The formalism we have described leaves room also for the "magnetic" Pauli operator (note that the wave function of the "sea" for the Magnetic Pauli operator depend on the magnetic field).

In presence of a smooth magnetic field one has still a Fermi sea but the wave functions are different. Diamagnetic inequalities imply that one has only a smooth modification of the minimal form.

If the field at the boundary is very intense one has a new minimal form: Gamma convergence selects a form that corresponds to different boundary conditions.

Since electrons *move on the boundary* any topological property can be derived from the wave functions at the boundary.

We have noticed that $\lambda \rightarrow \infty$ in \mathcal{M} corresponds to taking the semi-classical limit.

If the magnetic field is smooth at this scale the motion of electrons *on the surface of the Fermi sea* is seen as classical motion of point particles which satisfy the laws of *classical* electrodynamics [N].

In presence of very strong electromagnetic fields the Fermi surface in the semiclassical limit can have a non smooth structure and the description of dynamics may require a refined analysis [No] .

For a two-dimensional structure the domain occupied by conduction electrons is different.

It can no longer be approximated by a graph with a Y-shaped vertex and the approximation with contact interaction is no longer valid.

The number of bound states may be infinite but the picture of "Fermi sea" must be modified.

The conductivity may be higher even in presence of impurities.

XXXXXXXXXX

THE NELSON MODEL, THE NELSON POLARON

In the same way one can analyze the Nelson Model (strong contact of a particle of mass m with a quantum zero mass field).

In this case the Efimov structure reflects itself in the use of infinitely many representations of the zero mass field.

A quantum mechanical three body problem arises naturally if creation and annihilation operators are "partially dequantized" by choosing for *two of the zero mass particles* the ground state of a system in which the zero mass particles are in strong contact interaction with the massive one.

The ground state of the system is then obtained choosing for the remaining zero mass particles the vacuum state of a suitable representation of the c.c.r. The resulting ground state is a model for the *Nelson polaron* [G] , the ground state of the Nelson model [N] .

Notice that this procedure is limited to *linear couplings* of a particle and a quantized field.

We treat the Polaron problem in the context of second quantization. Second quantization can be thought as Weyl quantization for a system with an infinite number of particles. Lebesgue measure is substituted by a measure on function space (Gauss measure in the Bose case). Very roughly speaking in second quantization a wave function f is substituted with of a scalar field $\Psi(f) = a(f) + a^*(\bar{f})$ where $a(f)$ (resp. $a^*(\bar{f})$) destroys (resp. creates) a particle with wave $f \in L^2(\mathbb{R}^3)$. Both terms are linear in f

In the Bose case the field satisfies the (non relativistic) commutation relations $[\Psi(\bar{f}), \Psi(g)] = (f, g)$

One defines the *Fock representation* by postulating the existence of a vector Ω (the "vacuum") such that $a(f)\Omega = 0 \forall f$ in the Hilbert space. *Fock space* is the space generated by repeated action of the $a^*(f)$ on Ω (this justifies the name "creation operators")

A problem in the quantum theory of interacting quantum fields with coupling constant g is to find an irreducible representation (not necessarily Fock) in which the interaction is realized by operator-valued distribution.

We shall use the formalism of second quantization in a non relativistic setting and denote by $a(k)$ (resp. $a^*(k)$) the annihilation (resp. creation) of a zero mass particle " of momentum k " (we omit the more precise definition).

In the following we consider the strong contact interaction of the particle of mass m with two non relativistic zero mass particles.

We have already solved the three particle problem. The fact that the two identical particles have zero mass does not alter the procedure (the hamiltonian is still positive and the Krein map is well defined).

The formal expression of the potential is $-C(\delta(x - x_1) + \delta(x - x_2))$

The hamiltonian is the limit, in strong resolvent sense, of a sequence of hamiltonians with two body potentials that scale as $V_\epsilon(|x - y|) = \frac{1}{\epsilon^3} V\left(\frac{|x-y|}{\epsilon^3}\right)$

This system is called *polaronic* and the ground state is the *polaron* [N] .

We approximate the interaction by using the two-body potential $V^\epsilon = \frac{1}{\epsilon^3} V\left(\frac{|x_i - x|}{\epsilon}\right)$ where $V \in C^1$.

The free hamiltonian is

$$H_0 = -\frac{1}{2m} \Delta_x + \int \omega(p) a^*(p) a(p) dp \quad (8)$$

where $\omega(p) = |p|^2$ and the $a(k)$ satisfy the c.c.r.

We make use the formalism of second quantization *paying attention* to the fact that for zero mass particles there infinitely *inequivalent* representations of the c.c.r.

A vector of finite energy in the Hilbert space may contain an infinity of zero mass particles with smaller and smaller momentum (this is known as *infrared problem*).

We denote by \hat{H} the limit hamiltonian. It describes the strong contact of the massive particle with the two mass zero particles. The ground state of the system is called *polaron*.

To find the structure of the polaron we will "partially dequantize" the field by choosing properly the state *of two of the zero mass particles* (and therefore the representation of the c.c.r. since the zero mass particles are identical).

Let $\Psi \equiv \psi(x-x_1) \times \phi(x-x_2)$ be the ground state of the hamiltonian.

To find the structure of the ground state of the entire system we fiber the second quantization space of the zero mass particles *choosing as parameter the position of the particle of mass m* . We choose the representation by defining annihilation operators

$$A_x(y) = a(y) - \psi(x-y)f \tag{9}$$

For each value of x the (distribution valued) operators $A_x(y)$ satisfies the same c.c.r as the operators $a(y)$ but the two representations are *inequivalent*. Different values of the position of the particle of mass m correspond to a different "infrared behavior" of the mass zero field.

If one writes the Hamiltonian as a function of the field $A(y)$ one obtains

$$H = \hat{H} + \int \omega(p) A_x^*(p) A_x(p) dp$$

\hat{H} is an hamiltonian that describes the contact interaction of the massive particle with two identical zero mass particles . In the Theoretical Physics literature this operation goes under the name of "completing the square" and the particle of positive mass is now "dressed" with the a particles

To minimize the energy one chooses for each value of the coordinate x the vacuum (and therefore the Fock representation for $A_x(y)$).

Therefore the a particles are described in an x -dependent representation defined by

$$a_x(y) = A(y) + \Phi(x) \quad (10)$$

where the $A(p)$ is in the Fock representation.

There is no coupling. The ground state of the system has a cloud of mass zero A -particles.

The cloud *depends on the coordinate of the heavy particle*. [N] [F,S], [L,S] , [S]. This is known as *the infrared problem*.

One can also find all the excited states. Each excited state has a different cloud of zero mass particles.

Indeed the representation from which the vacuum is selected depends on the bound state of the three body problem that is chosen. It is easy to see that different states lead to inequivalent representations.

It is worth recalling that strong contact in a relativistic setting has a structure similar to the non relativistic case. In fact the relativistic metric compensates the fact that the "relativistic hamiltonian" is $\sqrt{-\Delta + m}$.

Also in a relativistic setting strong contact leads to an Efimov sequence of bound states and therefore to the presence of uncountably many representations of the c.c.r.

XXXXXXXXXX

STRONG CONTACT AS LIMITS

We prove that strong-contact hamiltonians are limit *in strong resolvent sense* of finite range hamiltonians. This makes contact hamiltonians a valuable tool in the study of interactions of very small range.

We require that the potential $V(|x|)$ be of class C^1 . It defines therefore a quadratic form in \mathcal{H}^1 .

By duality, it is a bounded map from \mathcal{H}^2 to C^1 (this explains why we find hamiltonians that are bounded below).

We consider separately the restriction of the forms to irreducible representation of the rotation group (the approximating potentials are invariant under rotation). The quadratic form associated to the potentials V^ϵ is a decreasing function of ϵ (the potential is negative).

Since there is no zero energy resonance the sequence of the approximating hamiltonians belongs to a compact subset for topology given by the Sobolev semi-norms.

The potential V is negative therefore for any choice of $V \in C^1 \cap L^1$ the ϵ -dependent quadratic forms are *strictly* decreasing as function of ϵ . A lower bound is the quadratic form of the contact interaction.

A decreasing sequence in a compact set with a lower bound admits always a converging subsequence. If the sequence is strictly decreasing the limit point is unique.

If the potential is of class C^1 the limit of this converging minimizing subsequence belongs to the limit set of the contact interactions.

Since this set contains only one element for any choice of the L^1 norm of the approximating potentials, the limit is unique and coincides with the contact interaction with the same strength.

Notice that, since the approach is variational, *no rate of convergence is obtained*.

Gamma convergence implies strong resolvent convergence [Dal].

Therefore the sequence of self-adjoint operators with potentials $H_0 + V^\epsilon$, $V^\epsilon \in C^1$ have in strong resolvent sense a limit which is the resolvent of the strong contact hamiltonian. In turn strong resolvent convergence implies convergence of spectra and of the Wave Operator in Scattering Theory.

We conclude the hamiltonian of a system describing the strong contact interactions of a particle with two identical bosons is limit, in the strong resolvent sense, of hamiltonians with two body *negative* rotationally invariant potentials of class C^1 that have support that shrinks to a point with law $V^\epsilon(|x|) = \frac{1}{\epsilon^3}V(\frac{|x|}{\epsilon})$.

The limit hamiltonian is bounded below. There are constant C_1, C_2 such that if $|V|_1 < C_1$ the negative spectrum is empty, if $C_1 \leq |V|_1 < C_2$ the strong contact hamiltonian has a finite negative spectrum while if $|V|_1 \geq C_2$ the negative spectrum is of Efimov type (the sequence of eigenvalues converges geometrically to zero).

In this latter case the eigenfunctions are centered on the barycenter of the system and have increasing support.

The same is true for *simultaneous pairwise weak contact* of three particles is the limit, in strong resolvent sense, of hamiltonians that have a zero energy resonance and potentials that scale with law $V^\epsilon(|x|) = \frac{1}{\epsilon^2} V(\frac{|x|}{\epsilon})$.

Weak contact of a particle with two identical particles is the limit of interactions with potential V^ϵ with this scaling. If the constant that measures the contact is large enough the limit hamiltonian has a bound state and no zero energy resonances.

In this case the Krein map helps in separating the small distance behavior from the long distance one.

Remark

One can prove that in three dimensions for $N \geq 3$ strong contact, weak-contact and interactions defined by regular Rollnik class potentials contribute *separately* and *independently* to the spectral properties and to the boundary conditions at the contact manifold.

For an unified presentation [D] (which includes also the proof that the addition of a regular potential does not change the picture) it is convenient to use a symmetric presentation due to Kato and Konno-Kuroda [KK] (who generalize previous work by Krein and Birman) for hamiltonians that can be written in the form $H = H_0 + H_{int}$ $H_{int} = B^*A$ where B, A are densely defined closed operators with $D(A) \cap D(B) \subset D(H_0)$ and such that, for every z in the resolvent set of H_0 , the operator $A \frac{1}{H_0 + z} B^*$ has a bounded extension, denoted by $Q(z)$.

Remark

In three dimensions for $N \geq 3$ strong contact, weak-contact and interactions defined by Rollnik type potentials contribute *separately* and *independently* to the spectral properties and to the boundary conditions at the contact manifold.

For an unified presentation (which includes also the proof that the addition of a regular potential does not change the picture) it is convenient to use a symmetric presentation due to Kato and Konno-Kuroda [KK] (who generalize previous work by Krein and Birman) for hamiltonians that can be written in the form $H = H_0 + H_{int}$ $H_{int} = B^*A$ where B, A are densely defined closed operators with $D(A) \cap D(B) \subset D(H_0)$ and such that, for every z in the resolvent set of H_0 , the operator $A \frac{1}{H_0 + z} B^*$ has a bounded extension, denoted by $Q(z)$.

References

[A] S.Albeverio,R. Hoegh-Krohn,Point Interaction as limits of short range interactions on Quantum Mechanics J. Operator Theory 6 (1981) 313-339

[A,S] K.Alonso, B.Simon The Birman-Krein-Visik theory of self-adjoint extensions. J.Operator Theory 4 (1980) 251-270 b

[A,L,M] C.Amovilli, F.Leys, H.March Electronic energy spectrum of two-dimensional solids.. J.Math. Chemistry 36 (2004)93-112

[B] M.Birman On the theory of self-adjoint extensions of positive definite operators Math. Sbornik. N.S. 38 (1956) 431-480 i

[[B,O,S] N.Benedikter, G. de Olivera, B.Schlein Derivation of the Gross-Pitaiewskii equation *Comm Pure Appl. Math* 68 (2015) 1399-1482

[B,P] H.Bethe-R.Peierls The scattering of Neutrons by Protons. *Proc of the Royal Soc. of London Series A* 149 (1935) 176-183

[B,D,G] L.Bruneau, J.Derezinski, V.Georgescu Homogeneous Schrödinger operators on a half-line *Ann. Inst. H.Poincaré* 12 (2011) 547-590

[Dal] G.Dal Maso Introduction to Gamma-convergence *Progr Non Lin. Diff Eq.* 8, Birkhauser (1993)

[D,R] S.Derezinsky, S.Richard On almost Homogeneous Schrödinger operators *arXiv* 1604 03340 (april 2016)

[E] V.Efimov Energy levels of three resonant interaction particles.
Nucler Physics A 210 (1971) 157-186

[E,S,Y] L.Erdos, B. Schlein, H T Yau Derivation of the Gross-Pitaevsii equation for the dynamics of a Bose-Einstein condensate
Ann. of Math. 172 (2010) 291-370

[K] M.G.Krein The theory of self-adjoint extensions of semi-bounded hermitiam transformations Math. Sbornik N.S. 20 (1962)1947
431-495

[K,K] R.Konno, S.T. Kuroda Convergence of of operators J, Fac,
Sci. Univ. Tokio 13 (1966) 55-63v

[K,S] M.Klaus, B.Simon Binding of Schrödinger particles through conspiracy of potential wells Ann. Inst. Henry Poincare XXX,
(1979) 83-98

[K.T] J.von Keler, S.Teufel The NLS limit of a particle in a wave guide arXiv 1510.03243

[L,O,R] A. Le Yaouanc, L.Oliver, J-C Raynal The Hamiltonian $(p^2 + m^2)^{\frac{1}{2}} - \frac{\alpha}{r}$ near the critical value $\alpha_c = \frac{2}{\pi}$. J Math. Phys. 38 (1997) 3997-4012)

[M1] R.Minlos On the point-like interaction between N fermions and another particle Moscow Math. Journal 11 (2011) 113-127

[M2] R.Minlos A system of three quantum particles with point-like interactions Russian Math. Surveys 69b (2014) 539-564

[Ma] K.A.Makarov Semiboundedness of the energy operator of three particle system wit delta-function interaction St. Ptersburg Math J 4 (1993) 967- 981

[M,F] R.Minlos, L.Faddeev On the point interaction for a three particle system in quantum mechanics Soviet Phys. Dokl. 6 (1962) 1072-1074

[M,M] K.A. Makarov, V.V. Melezhik Two sides of the coin :the Efimov effect and collapse. Teor. Math. Phys. 107 (1993) 755-769

[M,N] A.Ya Maltsev , S.P.Novikov The theory of closed 1-forms, level quasi-periodic functions and transport phenomena in electron systems. arXiv: 1805.05210

[N] E.Nelson Interaction of non-relativistic particle with quantized scalar fields J.Math.Phys. 5 (1964) 1190-1197

[No] S.P. Novikov Levels of quas-periodic function on the plane and Hamiltonian systems. Russian Math. Surveys 54 (1999) 1031-1032

[Pa] B.S.Pavlov Boundary conditions on thin manifolds Math. Sbornik 64 (1989) No 1

[Sh] Yu Shondin On the semi-boundedness of delta perturbations supported by curves with angle points Theor Math. Phys. 105 (1995) 1189-1200

[So] A.V.Sobolev The Efimov effect: discrete spectrum asymptotics Comm.Math.Phys.156 (1993) 101-126

[T] H.Tamura The Efimov effect of three body Schrödinger operators Journ. Functn. Analysis 95 (1991) 43-459

[T,S] K Ter-Martirosian , G Skorniakov Three body problem for short range forces Sov. Phys. JETP 4 (1956) 648-661

[V] M.Vishik On general boundary problems for elliptic differential operator Trudy Mos. Math. Obs 1 (1952) 187-246

[V, Z] S.A. Vugaltier, G.M. Zhislin The symmetry of the Efimov effect Comm.Math. Phys. 87 (1982) 89-103

[W,T] J. Wachs-muth, S.Teufel NLS in a narrow wave guide Mem-oirs of the A.M.S. 1083 (2013)

[Z] A.Zorich A problem of Novikov o the semi-classical motion of electrons in a plane Russian Math. Surveys 39 (1984) 287-288