Derivation of 1d and 2d Gross–Pitaevskii equations for strongly confined 3d bosons

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In a nutshell

Consider $N$ interacting bosons in a BEC which are in two or one spatial directions confined by a trap of diameter $\varepsilon$. 

Joint work with Stefan Teufel.

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We show that the dynamics of this system are effectively described by a one-/two-dimensional nonlinear equation.
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Microscopic model

Coordinates: \( z = (x, y) \in \mathbb{R}^3 \quad x \in \mathbb{R}^d \quad y \in \mathbb{R}^{3-d} \)

\( \mathcal{N} \)-body Hamiltonian

\[
H = \sum_{j=1}^{\mathcal{N}} \left( -\Delta_j + \frac{1}{\varepsilon^2} V^\perp \left( \frac{y_j}{\varepsilon} \right) \right) + \sum_{i<j} w_{N,\varepsilon}(z_i - z_j)
\]

- \( V^\perp \): confining potential; rescaled by \( \varepsilon \)
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Pair interaction:

\[
w_{N,\varepsilon}(z) := \mu^{1-3\beta} w \left( \mu^{-\beta} z \right) \quad \beta \in (0, 1]
\]

- \( w \geq 0 \) spherically symmetric, bounded, \( \text{supp} w \subseteq B_1(0) \)

- \( \mu = \left( \frac{N}{\varepsilon^{3-d}} \right)^{-1} \rightarrow \mu^{\beta} \): effective range of the interaction
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\(N\)-body Hamiltonian

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H = \sum_{j=1}^{N} \left( -\Delta_j + \frac{1}{\varepsilon^2} \nabla \left( \frac{y_j}{\varepsilon} \right) \right) + \sum_{i<j} w_{N,\varepsilon}(z_i - z_j)
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Limit:

\((N, \varepsilon) \rightarrow (\infty, 0)\) with suitable restrictions
Assumptions on the initial data

1. BEC:
   \[
   \lim_{(N,\varepsilon)\to(\infty,0)} \text{Tr} \left| \gamma_0^{(1)} - |\varphi_0^{\varepsilon}\rangle \langle \varphi_0^{\varepsilon}| \right| = 0
   \]

   - \( \gamma_0^{(1)} \): one-particle reduced density matrix of \( \psi_0^{N,\varepsilon} \)
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- **Condensate wave function:** \( \varphi_0^\varepsilon(z) = \Phi_0(x) \chi^\varepsilon(y) \)

- **transverse GS:** \( (-\Delta_y + \frac{1}{\varepsilon^2} V^\perp(\frac{y}{\varepsilon})) \chi^\varepsilon(y) = \frac{E_0}{\varepsilon^2} \chi^\varepsilon(y) \)
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- \(\Phi_0 \in H^{2d}(\mathbb{R}^d)\) \(\rightarrow\) evolves in time
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2. Energy per particle:
   \[ \lim_{(N,\varepsilon) \to (\infty,0)} \left| E(\psi_0^{N,\varepsilon}) - \mathcal{E}_{b_\beta}(\Phi_0) \right| = 0 \]

   - \( E(\psi) := \frac{1}{N} \langle \psi, H\psi \rangle - \frac{E_0}{\varepsilon^2} \)
   - \( \mathcal{E}_{b_\beta}(\Phi) := \langle \Phi, (-\Delta_x + \frac{1}{2} b_\beta |\Phi|^2) \Phi \rangle \)
Effective $d$-dimensional Gross–Pitaevskii dynamics

**Theorem**

Under assumptions (1) and (2) and for any $t \in \mathbb{R}$,

$$\lim_{(N, \varepsilon) \to (\infty, 0)} \text{Tr} \left| \gamma^{(1)}(t) - \langle \varphi^\varepsilon(t) \rangle \langle \varphi^\varepsilon(t) \rangle \right| = 0,$$

where $\varphi^\varepsilon(t) = \Phi(t) \chi^\varepsilon$ and $\Phi(t)$ is the solution of

$$i \frac{\partial}{\partial t} \Phi(t, x) = (-\Delta_x + b_\beta |\Phi(t, x)|^2) \Phi(t, x)$$

with $\Phi(0) = \Phi_0$ and where

$$b_\beta = \begin{cases} 
8\pi a \int |\chi(y)|^4 \, dy & \beta = 1 \quad \text{(GP)} \\
\|w\|_1 \int |\chi(y)|^4 \, dy & \beta \in (0, 1) \quad \text{(NLS)}
\end{cases}$$

$a$: scattering length of $w$, \quad $\chi$: ground state of $-\Delta_y + V^\perp(y)$
Related results

- X. Chen, J. Holmer. ARMA 2013.  \( d = 2 \)
  \( \rightarrow \beta \in (0, \frac{2}{5}) \), repulsive interactions

- X. Chen, J. Holmer. APDE 2017.  \( d = 1 \)
  \( \rightarrow \beta \in (0, \frac{3}{7}) \), attractive interactions

- J. v. Keler, S. Teufel. AHP 2016.  \( d = 1 \)
  \( \rightarrow \beta \in (0, \frac{1}{3}) \), repulsive interactions
Simultaneous limit \((N, \varepsilon) \to (\infty, 0)\)
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- **moderate confinement**: \(\varepsilon\) must not shrink too fast
  \(\rightarrow\) lower bound on \(\varepsilon\)
Parameter range for $\beta \in (0, 1)$

$d = 2$

\[
\beta = \frac{1}{3} \quad \varepsilon \quad N^{-1} \\
\beta = \frac{2}{3} \quad \varepsilon \quad N^{-1} \\
\beta = \frac{5}{6} \quad \varepsilon \quad N^{-1} \\
\beta = \frac{11}{12} \quad \varepsilon \quad N^{-1}
\]
Comparison with [ChHo2013] 

\[ d = 2, \beta \in (0, \frac{2}{5}) \]

- \( \beta = \frac{3}{11} \)
- \( \beta = \frac{11}{30} \)
- \( \beta = \frac{1}{3} \)
- \( \beta = \frac{23}{60} \)
Limiting sequences for $\beta = 1$ \hspace{1cm} d=2
## Strategy of proof

- **General strategy:** method from [Pickl2015]
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  - 3d micro dynamics $\leftrightarrow$ 1d/2d effective dynamics
  - split interaction into quasi-1d/2d interaction + remainders
  - remainders controllable with admissibility condition

Thank you very much for your attention!
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