Derivation of 1d and 2d Gross–Pitaevskii equations for strongly confined 3d bosons

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We show that the dynamics of this system are effectively described by a one-/two-dimensional nonlinear equation.

Joint work with Stefan Teufel.

References: 1d: J. Math. Phys. 60:031902; Ann. Henri Poincaré 20(3):1003
2d: arXiv:1907.04547

Microscopic model

Coordinates: $z = (x, y) \in \mathbb{R}^3$ $x \in \mathbb{R}^d$ $y \in \mathbb{R}^{3-d}$

N-body Hamiltonian

$$H = \sum_{j=1}^{N} \left(-\Delta_j + \frac{1}{\varepsilon^2} V^{\perp}\left(\frac{y_j}{\varepsilon}\right) \right) + \sum_{i<j} w_{N,\varepsilon}(z_i - z_j)$$

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Limit: $(N, \varepsilon) \to (\infty, 0)$ with suitable restrictions

Assumptions on the initial data

1. BEC: $$\lim_{(N,\varepsilon)\to(\infty,0)} \mathrm{Tr}\left|\gamma_0^{(1)} - |\varphi_0^\varepsilon\rangle\langle\varphi_0^\varepsilon|\right| = 0$$

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2. Energy per particle: $\lim_{(N,\varepsilon)\to(\infty,0)} \left|E(\psi_0^{N,\varepsilon}) - \mathcal{E}_{b_\beta}(\Phi_0)\right| = 0$

   - $E(\psi) := \frac{1}{N}\langle\psi, H\psi\rangle - \frac{E_0}{\varepsilon^2}$
   - $\mathcal{E}_{b_\beta}(\Phi) := \langle\Phi, \left(-\Delta_x + \frac{1}{2}b_\beta|\Phi|^2\right)\Phi\rangle$

Effective d-dimensional Gross–Pitaevskii dynamics

Theorem

Under assumptions (1) and (2) and for any $t \in \mathbb{R}$,

$$\lim_{(N,\varepsilon)\to(\infty,0)} \mathrm{Tr}\left|\gamma^{(1)}(t) - |\varphi^\varepsilon(t)\rangle\langle\varphi^\varepsilon(t)|\right| = 0,$$

where $\varphi^\varepsilon(t) = \Phi(t)\chi^\varepsilon$ and $\Phi(t)$ is the solution of

$$\mathrm{i}\frac{\partial}{\partial t}\Phi(t,x) = \left(-\Delta_x + b_\beta|\Phi(t,x)|^2\right)\Phi(t,x)$$

with $\Phi(0) = \Phi_0$ and where

$$b_\beta = \begin{cases} 8\pi a \int |\chi(y)|^4\,\mathrm{d}y & \beta = 1 \qquad \text{(GP)} \\ \|w\|_1 \int |\chi(y)|^4\,\mathrm{d}y & \beta \in (0,1) \qquad \text{(NLS)} \end{cases}$$

a: scattering length of w, χ: ground state of $-\Delta_y + V^\perp(y)$

Related results

- X. Chen, J. Holmer. ARMA 2013. $d = 2$

  $\rightarrow \beta \in (0, \frac{2}{5})$, repulsive interactions

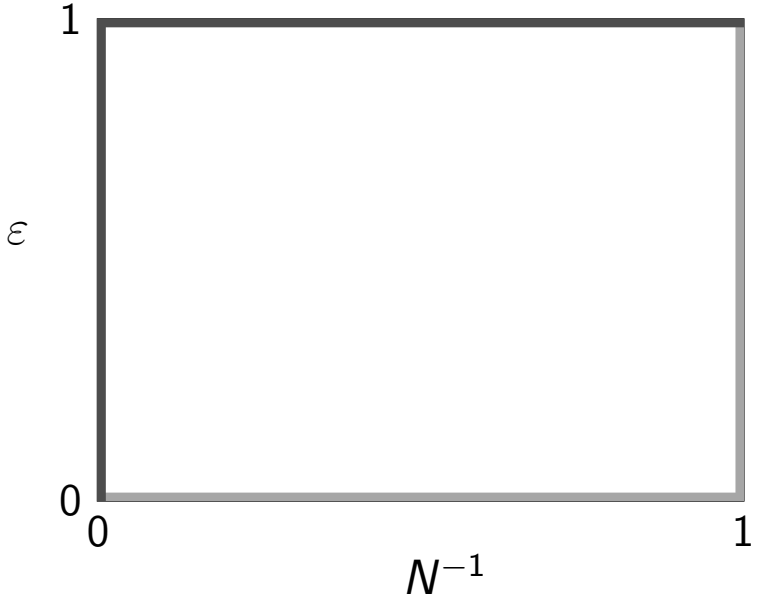
- X. Chen, J. Holmer. APDE 2017. $d = 1$

  $\rightarrow \beta \in (0, \frac{3}{7})$, attractive interactions

- J. v. Keler, S. Teufel. AHP 2016. $d = 1$

  $\rightarrow \beta \in (0, \frac{1}{3})$, repulsive interactions

Simultaneous limit $(N, \varepsilon) \to (\infty, 0)$



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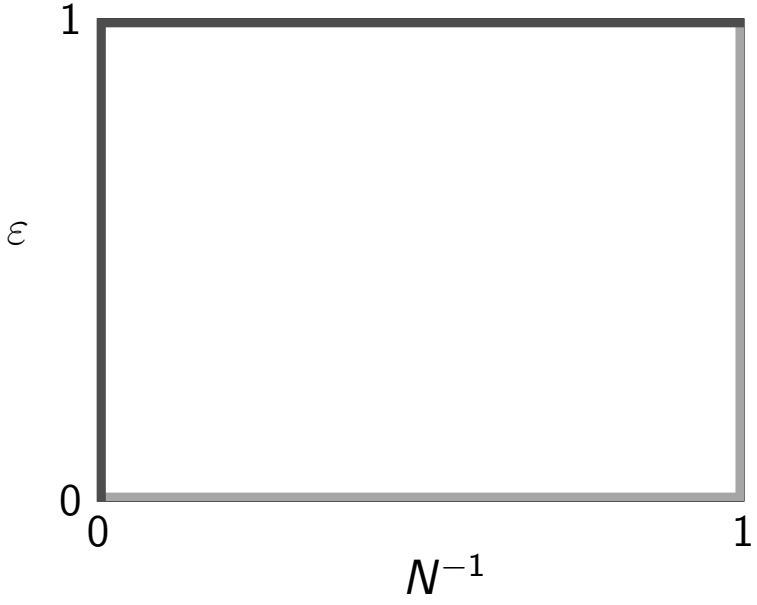

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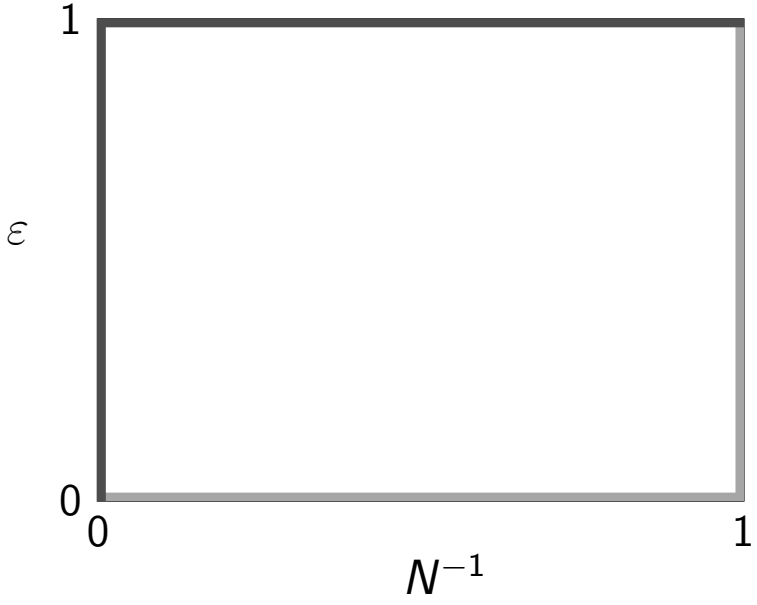


- admissibility condition: ε must shrink fast enough

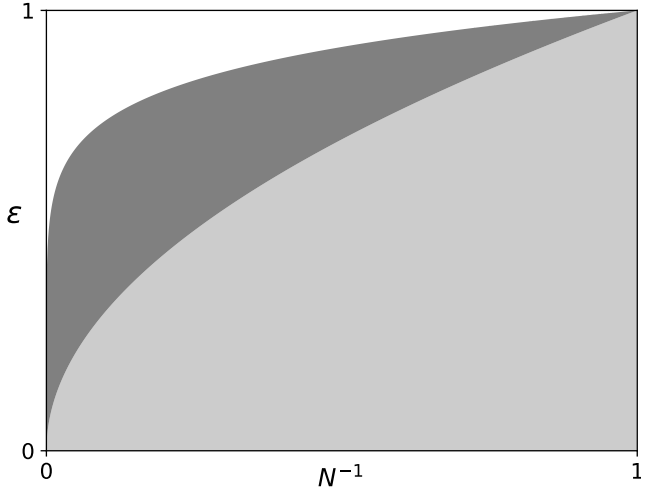
  $\to$ upper bound on ε

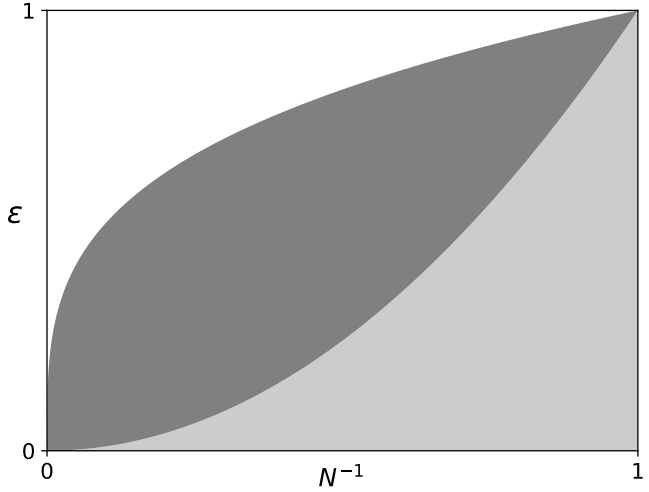
- moderate confinement: ε must not shrink too fast

  $\to$ lower bound on ε

Parameter range for $\beta \in (0,1)$ d=2

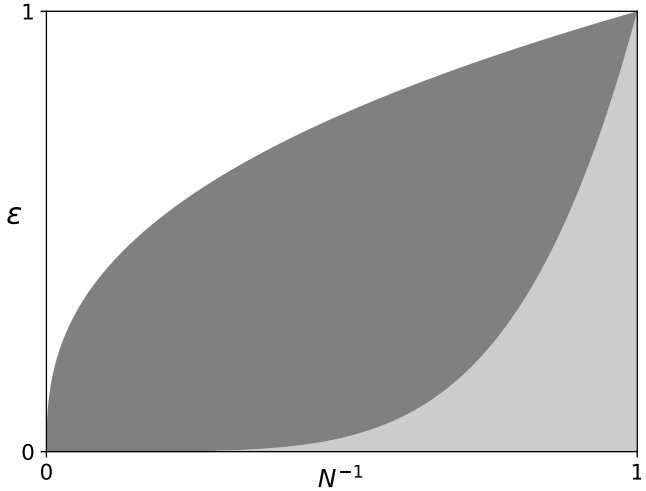
$\beta = \frac{1}{3}$

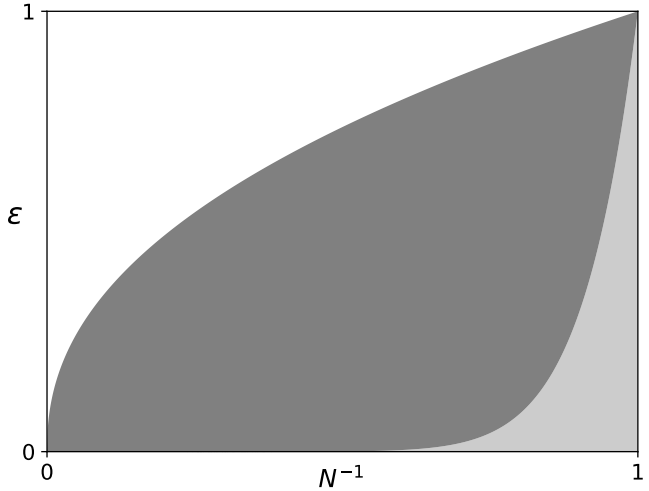




$\beta = \frac{2}{3}$

$\beta = \frac{5}{6}$



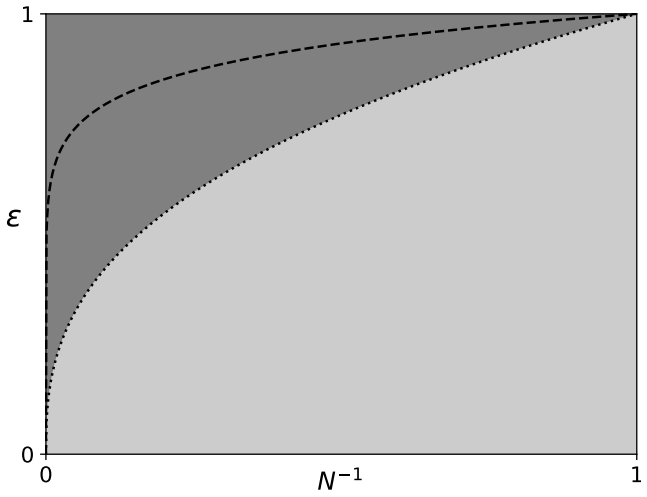


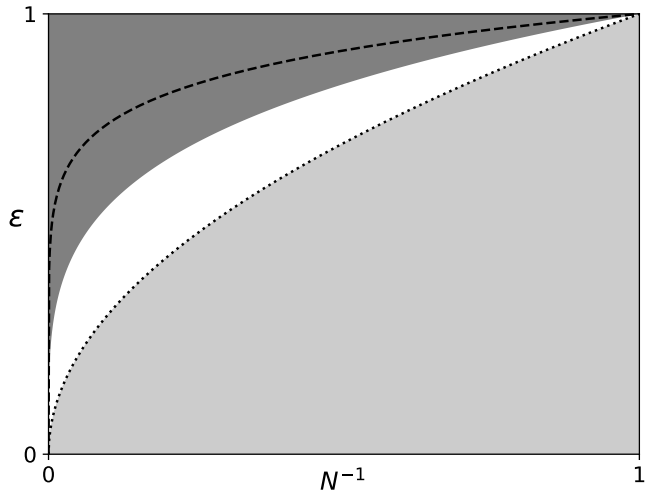
$\beta = \frac{11}{12}$

Comparison with [ChHo2013]

$d = 2,\ \beta \in (0, \frac{2}{5})$

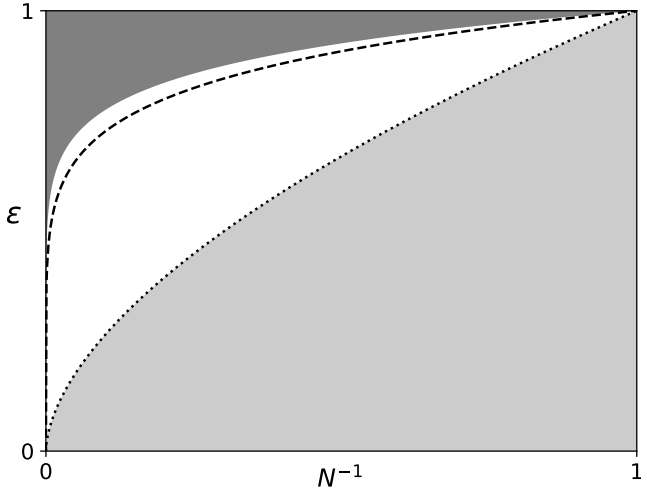
$\beta = \frac{3}{11}$

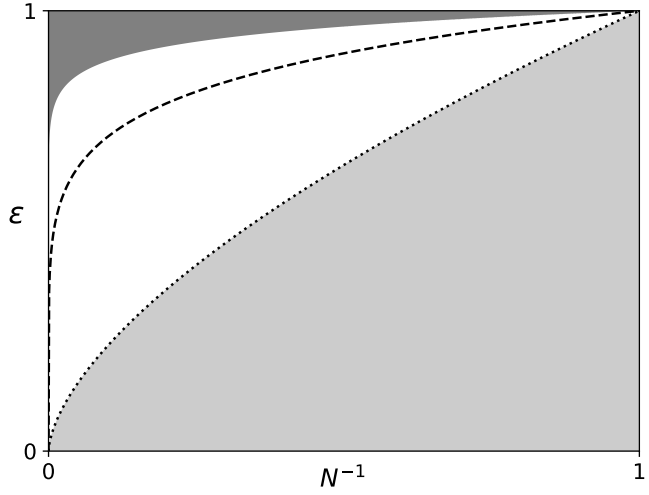




$\beta = \frac{1}{3}$

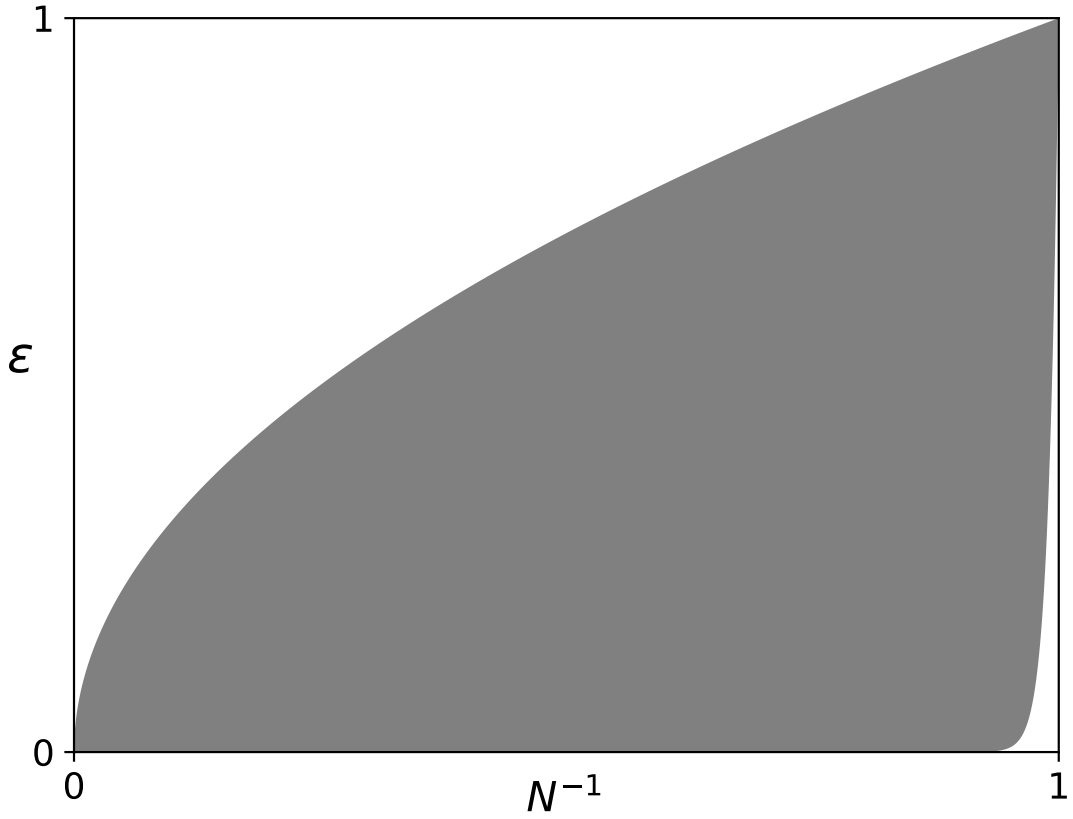
$\beta = \frac{11}{30}$





$\beta = \frac{23}{60}$

Limiting sequences for $\beta = 1$ d=2



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Thank you very much for your attention!