The phase transition for random loop models on trees

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Joint work with Johannes Ehlert, Benjamin Lees, Lukas Roth
The random loop model: intuition
The random loop model: definition

- \( G = (V, E) \) a graph. Parameters \( \beta, u \).
- \( (X_e^X)_{e \in E} \) iid PPP with intensity \( u \) on \([0, \beta)\) ('crosses').
- \( (X_e^\|)_{e \in E} \) iid PPP with intensity \( 1 - u \) on \([0, \beta)\) ('bars').
- \( \mathbb{T}_\beta \) torus, \( X = \{(v, t) : v \in V, t \in \mathbb{T}_\beta\} \).
- The set \( \bigcup_{e: v \in e} X_e^X \cup X_e^\| \) separates \( \{(v, t) : t \in \mathbb{T}_\beta\} \) into disjoint open intervals. \( U(v, t) \) is the interval containing \( t \).
- Connections: \( (v, t) \sim (v', t') \) if
  - \( v = v' \) and \( t' \in U(v, t) \), or
  - \( e := \{v, v'\} \in E \), and there is precisely one element of \( X_e^X \) (and none of \( X_e^\| \)) between \( t \) and \( t' \) (considering periodicity) or
  - \( e := \{v, v'\} \in E \), \( U(v, t) \cap U(v', t') \neq \emptyset \) and has at least one boundary point in \( X_e^\| \).
  - Extend by transitivity.
- Percolation type model. **Question**: infinite cluster?
The lack of monotonicity

The main difficulty: adding connections can decrease the size of a connected component. Two mechanisms:

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We are only able to address problem 1.
Random loop model and quantum theory

- Let \( P_{u,\beta} \) be the joint measure of the PPP \( (X^X_e)_{e \in E}, (X^\parallel_e)_{e \in E} \).
- For \( \theta > 0 \), \( G \) finite let \( P_{\theta,u,\beta}(A) = \frac{1}{\mathbb{E}_{u,\beta}(\theta^L)} \mathbb{E}_{u,\beta}(\theta^L 1_A) \), where
- \( L(\omega) \) is the total number of loops in the configuration produced by the \( X^X_e,\parallel(\omega) \).
- Relevant quantum system has Hamiltonian

\[
H = -2 \sum_{\{x,y\} \in E} S_x^{(1)} S_y^{(1)} + S_x^{(2)} S_y^{(2)} + (2u - 1) S_x^{(3)} S_y^{(3)}.
\]

- Heisenberg ferromagnet \( (u = 1) \), anti-ferromagnet \( (u = 0) \) or \( XY \)-model \( (u = 1/2) \).
- Example for connection to random loop models:

\[
\langle S_x^{(1)} S_y^{(1)} \rangle_\beta \equiv \frac{\text{tr}(S_x^{(1)} S_y^{(1)} e^{-\beta H})}{\text{tr} e^{-\beta H}} = P_{2,u,\beta}(x \leftrightarrow y).
\]
History of the random loop model

Case $\theta = 2$:

▶ Feynman 1953: basic idea to treat thermal states using functional integrals.
▶ Conlon and Solovej 1991: random walk representation for the ferromagnet.
▶ Toth 1993 improves this result using a random loop model.
▶ Aizenman and Nachtergaele 1994: extension to more general spin values and interactions.
▶ Ueltschi 2013: extension to general $\theta$ and all $u$. 

Case $\theta = 1$:

▶ Harris 1972: random stirring model.
▶ Schramm 2005: emergence of infinite cycles for the complete graph.
▶ Kotecky, Milos, Ueltschi (2016), results on the hypercube.

V. Betz (Darmstadt) Loops on trees
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Random loop models on trees

- Angel 2003: proof of existence of long loops for $d \geq 4, \theta = 1$.
- Hammond 2013: sharp phase transition in $\beta$ for $d \geq 55$, for $u = 1, \theta = 1$.
- Hammond 2015: strict bounds on $\beta_c$ for very high $d$.
- Hammond and Hedge 2018: improved those bounds to $d \geq 56$ and general $u$.
- Björnberg, Ueltschi 2018, 2019: Asymptotics for large $d$, and for all $\theta, u$:

$$\frac{\beta_c}{\theta} = \frac{1}{d} + \frac{1 - \theta u (1 - u) - \theta^2 (1 - u)^2 / 6}{d^2} + o(d^{-2}).$$

- Topic of this talk: proof of sharp phase transition for $\theta = 1$ and all $d \geq 3$, and (in principle) full asymptotic expansion of $\beta_c$. 

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**Main Theorem**

**Theorem:** $T$ the infinite $d$-regular, $r$ its root, $\gamma_T$ the loop $(r, 0)$.

For all $d \geq 3$, $u \in [0, 1]$ there exists $\beta_c > 0$ and $\beta^+ > \beta_c$ such that

1. $\gamma_T$ is finite almost surely for all $\beta \leq \beta_c$,

2. $\gamma_T$ is infinite with positive probability for all $\beta \in (\beta_c, \beta^+)$.

Moreover, $\beta^+ \geq \frac{1}{\sqrt{d}}$ for all $d \geq 3$ and $\beta^+ = \infty$ for $d \geq 16$. 
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Moreover, $\beta^+ \geq \frac{1}{\sqrt{d}}$ for all $d \geq 3$ and $\beta^+ = \infty$ for $d \geq 16$.

We have the expansion

$$\beta_c = \sum_{k=0}^{n} \frac{\alpha_k(u)}{d^{k+1}} + O(d^{-n-2}),$$

(1)

where the $\alpha_k$ are polynomials of order $2k$ in $u$ with recursively computable coefficients.
Main idea of the proof: Galton Watson trees

Let $C_1$ be the (random) maximal subtree of $T$ containing the root and where each edge has at least two links.

Percolation on trees: $C$ is finite almost surely if $\beta^2 \leq 1/d$. 

$\gamma_T$ in both directions or not at all.

So, $e$ serves as a renewal edge, separating future and past.

Let $M_i$ be the 'living' renewal edges in the $i$-th generation. $(|M_i|)$ for $i \in \mathbb{N}$ is a Galton-Watson process.

Therefore: $\gamma_T$ is finite almost surely if and only if $E \mu, \beta(M_1) \leq 1$. 

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- either it also leaves the next (iid) subtree $C_2$ attached at $y$
- or it returns by the same way into $C_1$, i.e. using $e$ in the opposite direction.

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- Let $M_i$ be the 'living' renewal edges in the $i$-th generation. $\left(|M_i|\right)_{i \in \mathbb{N}}$ is a Galton-Watson process.

- Therefore: $\gamma_T$ is finite almost surely if and only if $\mathbb{E}_{u,\beta}(M_1) \leq 1.$
Computing $\mathbb{E}_{u,\beta}(M_1)$

Let $C$ be the set of finite rooted subtrees of $T$, with edges $e$ labelled by $n_e \geq 2$. In the case $\beta < 1/\sqrt{d}$ we find

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\mathbb{E}(|M_1|) = \sum_{C \in \mathcal{C}} \mathbb{E}(|M_1| | C_1 = C) \mathbb{P}(C_1 = C).
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By independence, we have

$$\mathbb{P}(C_1 = C) = \prod_{e \in E(C)} \mathbb{P}(|X_e| = n_e) \prod_{e \in \partial^+ C} \mathbb{P}(|X_e| \leq 1)$$

$$= \frac{\beta^N(C)}{\prod_{e \in E(C)} n_e!} e^{-\beta d|V(C)|} (1 + \beta)^{(d-1)|V(C)|+1}.$$

and $\mathbb{E}(|M_1| | C_1 = C) = \frac{\beta}{1+\beta} \sum_{x \in V(C)} (d - d_x) p(u, d)$ where $p(u, d)$ is a polynomial in $u$. 
Uniqueness and sharpness of the phase transitions

The end result is

$$\mathbb{E}_\beta(|M_1|) = \sum_{C \in \mathcal{C}} f_C(\beta d, d^{-1}) g_C(d^{-1}, u)$$

where (with $N(C)$ the number of links on $C$)

$$f_C(\alpha, h) = \left( e^{-\alpha} (1 + \alpha h)^{1/h-1} \right)^{|V(C)|} \frac{\alpha^{N(C)+1}}{(N(C) + 1)!} h^{N(C) - E(C)}.$$

and $g_C(h, u)$ a polynomial, non-negative for all $u$ and all $h = 1/d, \ d \in \mathbb{N}$.
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and $g_C(h, u)$ a polynomial, non-negative for all $u$ and all $h = 1/d$, $d \in \mathbb{N}$. Now a direct computations shows that

$$\partial_\alpha f_C(\alpha, h) = \left( -|V(C)| + \frac{(1/h - 1)|V(C)| h}{1 + \alpha h} + \frac{N(C) + 1}{\alpha} \right) f_C(\alpha, h)$$

and since $N(C) + 1 \geq 2|E(C)| + 1 \geq |V(C)|$, we find that

$$\partial_\alpha f_C(\alpha, h) \geq f_C(\alpha, h)|V(C)| \frac{1 - \alpha^2 h}{(1 + \alpha h)\alpha} > 0.$$

for $\alpha^2 h = \beta^2 d < 1$, thus $\beta \mapsto \mathbb{E}_\beta(|M_1|)$ is strictly monotone.
Existence of the phase transition

\[ E_{\beta}(|M_1|) = \sum_{C \in C} f_C(\beta d, d^{-1}) g_C(d^{-1}, u) \]

This is a sum of positive (explicit) terms, so by taking enough of them, we can achieve \( E_{\beta}(|M_1|) > 1 \) for large enough \( \beta \).
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For \( d \geq 16 \), these region overlap with the non-reentry regions of Hammond 2015 \( \Rightarrow \) unique phase transition.
Expanding $\beta_c$ in powers of $1/d$.

- Put $\beta = \alpha/d$ and sort the elements of $C$ by their contribution in the limit $d \to \infty$ at fixed $\alpha \approx 1$.

- Approximate

$$f(\alpha) = \mathbb{E}(|M_1|) = \sum_{n=1}^{\infty} \sum_{C \in C_n} \mathbb{E}(|M_1|_{C_1=C})$$

from below: sum only up to order $N$.

- Approximate $\mathbb{E}(|M_1|)$ from above by making all loops in $C_M$, $M > N$ survive to the boundary.

- This gives two analytic functions

$$f_-(\alpha) < f(\alpha) < f_+(\alpha)$$. The Ansatz $f_-(\alpha^+_c(d)) = 1$, $f_+(\alpha^-_c(d)) = 1$ yields analytic upper and lower bounds for $\alpha_c(d)$.

- They agree up to order $N$ in $1/d$. 
An intriguing observation

We calculated $\beta_c$ up to sixth order. The result is

$$\beta_c(u, d) = \sum_{k=0}^{5} \frac{\alpha_k(u)}{d^{k+1}} + O(d^{-7}), \quad (2)$$

with $\alpha_k(u) = \sum_{j=0}^{2k} \alpha_{k,j} \binom{2k}{j} u^j (1-u)^{2k-j}$, and

<table>
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<th>$\alpha_{k,j}$</th>
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<th>$k = 2$</th>
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<tbody>
<tr>
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<tr>
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<tr>
<td>$j = 5$</td>
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<tr>
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So, in the Bernstein basis, all coefficients seem to be in $[0, 1]$. We have no idea why this is so and whether it persists.
Conclusion

- For the $d$-regular tree, the loop model with $\theta = 1$ is now very well understood.
- The case $\theta > 1$ can’t use renewal theory directly, but is not hopeless → work in progress.
- The real challenge is to understand any model with finite degree, and where loops in the graph play an essential role.
- A mystery remains in the peculiar properties of the coefficients of $\beta_c(u, d)$.

Thank you for your attention!