

Marginal triviality of the scaling limits of critical 4D Ising and ϕ_4^4 models

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Talk based on a joint work with Hugo Duminil-Copin.

Quantissima in the Serenissima
Venice, 22 Aug. 2019.

Different perspectives and a common statement

Continuum field theories with local interaction are of interest from a number of perspectives:

1. *Field theory*: This concept plays a basic role in the physics discourse, ranging from quantum field theory to condensed matter physics.

Field theory's mathematical formulation poses difficulties which so far have been only partially addressed. Beyond Gaussian fields, which exist in any dimension, "non-trivial" field theories have been constructed over \mathbb{R}^2 and \mathbb{R}^3 . However it was also proved that the approach used there does not yield the desired result for $d > 4$ dimensions (Aiz. '81, Frölich '82).

Partial results have indicated that the same may hold true for the critical dimension $d = 4$, however a sweeping statement such as proved for $d > 4$ has remained open. In this work we address this case.

2. Functional integration – the Euclidean version of the above challenge.

Make sense of probability measures on the space of distributions over \mathbb{R}^d , of the form

$$\langle F(\phi) \rangle = \int F(\phi) e^{-\int_{\mathbb{R}^d} \lambda(P(\phi(u)) + J|\nabla\phi|^2(u)) du} \prod_{x \in \mathbb{R}^d} d\phi(x) / \text{Norm}$$

with $P(\phi)$ a polynomial, of which the lowest order even deviation from the trivial case would be $P(\phi) = \lambda\phi(u)^4 + B\phi^2$ (with B possibly negative).

One can spot a number of problems with this informal expression.

Partially successful attempts at their resolution has been the focus of substantial body of work, employing different means (regularizing cutoffs, scale decomposition, renormalization group flows, the theory or regularity structures, etc.), with results like those mentioned above.

Beyond one dimension such measures are not support on continuous functions. Basic measurable quantities are functionals

$$T_g(\phi) := \int_{\mathbb{R}^d} g(x)\phi(x) dx,$$

associated with $g \in C_0(\mathbb{R}^d)$. Expectation values of products take the form:

$$\langle \prod_{j=1}^n T_{g_j}(\phi) \rangle = \int_{\mathbb{R}^n} dx_1 \dots dx_n S_n(x_1, \dots, x_n) \prod_{j=1}^n g(x_j)$$

with the correlation functions $S_n(x_1, \dots, x_n)$,

a.k.a the field theory's Schwinger functions.

The correlation functions encode properties of the field such as translation invariance, reflections positivity, and various characteristic exponents.

In the cases under considerations here odd correlation functions vanish, and $S_2(0, x)$ is locally integrable.

The variables $T_g(\phi)$ are then jointly Gaussian (of zero mean) and variance readable from $S_2(x_1, x_2)$ if and only if the field's correlation functions satisfy Wick's law:

$$S_n(x_1, \dots, x_{2n}) = \sum_{\Pi} \prod_{j=1}^n S_2(x_{\Pi(2j-1)}, x_{\Pi(2j)}) := \mathcal{G}_n[S_2](x_1, \dots, x_{2n})$$

the sum being over pairing permutations.

Such field theories are colloquially referred to as **trivial**; their n point functions are given by a simple functional of two point function, and a relatively simple admissibility test is required of S_2 .

As mentioned above the challenge to construct non-trivial examples was met in dimensions $d < 4$. For $d > 4$ we have negative results (which Alan Sokal termed “deconstructive field theory”).

J. Glimm and A. Jaffe, *Positivity of the ϕ_3^4 hamiltonian.*, Forts. der Physik '73.

F. Guerra, L. Rosen and B. Simon, *The $P(\phi)_2$ Euclidean Quantum Field Theory as Classical Statistical Mechanics*, Ann. Math. '75.

M. Aizenman *Proof of the Triviality of ϕ_d^4 Field Theory and Some Mean-Field Features of Ising Models for $d > 4$* , PRL '81.

M. Aiz., *Geometric analysis of φ^4 fields and Ising models. I, II*, CMP '82.

J. Fröhlich, *On the triviality of $\lambda\phi_d^4$ theories and the approach to the critical point in $d > 4$ dimensions*, Nuc. Phys. '82.
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D. Brydges, J. Fröhlich, and T. Spencer, *The random walk representation of classical spin systems and correlation inequalities*, CMP '82.

C. Gawedzki, A. Kupiainen, *Massless lattice ϕ_4^4 theory: Rigorous control of a renormalizable asymptotically free model*, CMP '85.

H. Tasaki and T. Hara, *A rigorous control of logarithmic corrections in four-dimensional ϕ^4 spin systems*, JSP '87.

R. Bauerschmidt, D.C. Brydges, and G. Slade, *Scaling limits and critical behavior of the 4-dimensional n -component $|\phi^4|$ spin model*, JSP '14.

Note: the papers in this list addressing ϕ_4^4 focused on weakly non-quadratic interactions.

Statistical mechanics:

Euclidean field theory is of relevance for the theory of critical phenomena. A guiding example is provided by the Ising model on \mathbb{Z}^d , with ($J > 0$)

$$H(\sigma) = J \sum_{\{u,v\}; |u-v|=1} |\sigma_u - \sigma_v|^2 / 2 - h \sum_x \sigma_x.$$

The system's Gibbs equilibrium states are probability measures defined by

$$\langle F(\sigma) \rangle = \sum_{\sigma} F(\sigma) e^{-\beta H(\sigma)} / \text{Norm}.$$

For any $\beta \neq \beta_c$: $\langle \sigma_{x_1}; \sigma_{x_2} \rangle_{\beta} := \langle \sigma_{x_1} \sigma_{x_2} \rangle - \langle \sigma_{x_1} \rangle \langle \sigma_{x_2} \rangle \leq \mathbf{A}(\beta) e^{|x_1 - x_2| / \xi(\beta)}$

However, at β_c that changes to power-law decay. It is then of interest to

consider $\tau_{g;\ell,\beta}(\sigma) := \alpha(\ell) \sum_{u \in \mathbb{Z}^d} g(u/\ell) \sigma_u$

with $g \in C_0(\mathbb{R}^d)$, and $\alpha(\ell)$ selected so that the second moments are bounded above and below, uniformly in $\ell < \infty$. Then

$$\left\langle \prod_{j=1}^n \tau_{g_j;\ell,\beta}(\sigma) \right\rangle = \int dx_1 \dots dx_n S_{n;\ell,\beta}(x_1, \dots, x_n) \prod_{j=1}^n g(x_j) [1 + o(\frac{1}{\ell})]$$

with the rescaled correlation functions

$$S_{n;\ell,\beta}(x_1, \dots, x_n) = \alpha(\ell)^n \left\langle \prod_{j=1}^n \sigma_{\lfloor x_j \ell \rfloor} \right\rangle_{\beta}$$

For the Ising model we prove that in $d = 4$ dimensions, in the limit $\ell \rightarrow \infty$ the above variables are jointly Gaussian, in the sense that their correlation functions asymptotically obey Wick's law:

Theorem: There exists $C_n > 0$ such that for every $x_1, \dots, x_{2n} \in \mathbb{R}^4$,

$$\left| S_{\ell, \beta_c}(x_1, \dots, x_{2n}) - \sum_{\pi \text{ pairing}} S_{\ell, \beta_c}(x_{\pi(1)}, x_{\pi(2)}) \cdots S_{\ell, \beta_c}(x_{\pi(2n-1)}, x_{\pi(2n)}) \right| \leq \frac{C_n}{(\log L)^c} S_{\ell, \beta_c}(x_1, \dots, x_{2n}).$$

A similar bound holds for Φ_4^d functional intervals with lattice cutoff, uniformly in the model's bare parameters.

The proof for $d > 4$ relied heavily on:

Proposition 1 The infrared bound (GJ '73, FSS '76): for the model on \mathbb{Z}^d

$$\langle \sigma_0 \sigma_x \rangle_{\beta_c} \leq C_d / |x|^{d-2} \quad \text{with } C_d < \infty \text{ for any } d > 2.$$

Proposition 2 The tree diagram bound (Aiz. '82)

For the Ising model on any finite graph, and any $\beta \geq 0$

$$u_4^{(\beta)}(x_1, x_2, x_3, x_4) := \langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sigma_{x_4} \rangle - \\ [\langle \sigma_{x_1} \sigma_{x_2} \rangle \langle \sigma_{x_3} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_3} \rangle \langle \sigma_{x_2} \sigma_{x_4} \rangle + \langle \sigma_{x_1} \sigma_{x_4} \rangle \langle \sigma_{x_2} \sigma_{x_3} \rangle]$$

satisfies

$$|u_4(x, y, z, t)| \leq 2 \sum_u \langle \sigma_x \sigma_u \rangle \langle \sigma_y \sigma_u \rangle \langle \sigma_z \sigma_u \rangle \langle \sigma_t \sigma_u \rangle$$

For a heuristic explanation of the criticality of $d = 4$ notice that, at a somewhat sloppy level of estimate, for a quadruple of points at distances of order L the bound on u_4 differs from the full 4-point function $\langle \sigma_{x_1} \sigma_{x_2} \sigma_{x_3} \sigma_{x_4} \rangle$ by: i) a volume factor of L^d , due to the summation over U , ii) two pair correlations which by the infrared bound do to exceed C/L^{d-2} .

This suggests that

$$|U_4^{(\beta)}(x_1, x_2, x_3, x_4)| \leq \frac{1}{L^{d-4}} S_4^{(\beta)}(x_1, x_2, x_3, x_4) \quad \text{where } \frac{1}{L^{d-4}} = \frac{L^{d-4}}{L^{2(d-2)}}$$

The above indicates that in $d > 4$ dimensions, in the limit $L \rightarrow \infty$, Wick's relation holds at least for the 4-point function. The implication then extends to all orders through the following general result.

Proposition 3 Bounds for higher moments (Aiz. '82):
For the Ising model on any finite graph

$$0 \leq -[S_{n;\ell,\beta}(x_1, \dots, x_n) - \mathcal{G}_n[S_{2;\ell,\beta}](x_1, \dots, x_n)]$$

$$\leq \frac{3}{2} \sum_{1 \leq i < j < k < l \leq n} S_{n;\ell,\beta}(x_1, \dots, \cancel{x_i}, \dots, \cancel{x_j}, \dots, \cancel{x_k}, \dots, \cancel{x_l}, \dots, x_n) |U_4(x_i, x_j, x_k, x_l)|$$

with
$$\mathcal{G}_n[S_2](x_1, \dots, x_{2n}) := \sum_{\Pi} \prod_{j=1}^n S_2(x_{\Pi(2j-1)}, x_{\Pi(2j)}) .$$

The new result which enables to extend the triviality of the scaling limit to $d = 4$ is based on an improvement of the tree diagram bound by the factor $C/[\log L]^c$. This is accomplished through a version of multi-scale analysis.

For further discussion we move to the board

Thank you for your attention.