Nodal sets and geometric control

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Nodal intersection problems

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Let (M^m, g) be a compact *real analytic* Riemannian manifold without boundary of dimension m. We denote by $\{\varphi_j\}_{j=0}^{\infty}$ an orthonormal basis of Laplace eigenfunctions,

$$-\Delta \varphi_j = \lambda_j^2 \varphi_j, \quad \langle \varphi_j, \varphi_k \rangle = \delta_{jk},$$

where $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and where $\langle u, v \rangle = \int_M uv dV_g$ (dV_g being the volume form).

We denote the nodal set of an eigenfunction φ_λ of eigenvalue $-\lambda^2$ by

$$\mathcal{N}_{\varphi_\lambda} = \{x \in M : \varphi_\lambda(x) = 0\}.$$

Sharp upper bounds for $\mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda})$ were proved by Donnelly-Fefferman in the 80's in the real analytic case:

$$c\lambda \leq \mathcal{H}^{m-1}(\mathcal{N}_{\varphi_\lambda}) \leq C\lambda.$$

Restrictions of eigenfunctions to a submanifold

Let $H \subset M$ be a real analytic submanifold. Much work has gone into the study of the restrictions $\varphi_j|_H$, its norms and its zeros.

Let $\mathcal{S} = \{j_k\}_{k=1}^{\infty}$ be a subsequence (indices of) eigenvalues. (We also let $\mathcal{S}$ denote $\{\lambda_{j_k}\}$ or the sequence $\{\varphi_{j_k}\}$ of eigenfunctions from the given orthonormal basis.)

Question: For curves or hypersurfaces, estimate the Hausdorff measure of $\mathcal{N}_{\varphi_\lambda} \cap H$ $=$ nodal set of $\varphi_{j_k}|_H$.

Extreme cases

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The answer depends on dim H at least and we mainly consider dim $H = 1$ (curve) or dim $H = m - 1$ (hypersurface).

If $H = \mathrm{Fix}(\sigma)$ is the fixed point set of an isometric involution $\sigma : M \to M$, then H can be a hypersurface (e.g. $x_n \to -x_n$ on S^{n-1} or on $\mathbb{R}^n$) or of lower dimension (e.g. the fixed point set of $Z \to \bar{Z}$ on $\mathbb{C}^m$ is totally real $\mathbb{R}^m$).

Odd eigenfunctions vanish on $\mathrm{Fix}(\sigma)$. I.e. 'half' of all eigenfunctions vanish on this set.

Bourgain-Rudnick: Characterize $H \subset M$ such that there exists *some* infinite sequence $\mathcal{S}$ such that $\varphi_{j_k}|_H = 0$. We call such submanifolds 'nodal'.

We are able to answer the question for subsequences of positive number density.

‘$\mathcal{S}$-Good submanifolds’

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A less restrictive condition than nodal is ‘$\mathcal{S}$-bad’: It means that $\sup_{x \in H} |\varphi_{j_k}|_H \leq Ce^{-M\lambda_{j_k}}$ for all $M > 0$. “Super-exponential decay’.

We say that H is $\mathcal{S}$- ‘good’ if there exists $M > 0$ so that

$$\sup_{x \in H} |\varphi_{j_k}|_H \geq Ce^{-M\lambda_{j_k}}.$$

A good curve

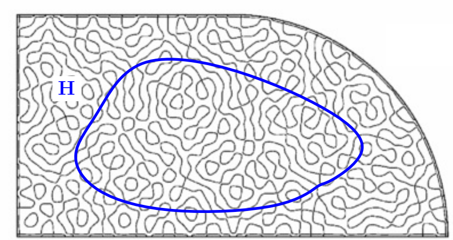

Figure 1: Nodal lines of a high energy state, $\lambda \sim 84$, in the quarter stadium.

Main results in a nutshell

We prove that the number $n(\varphi_{j_k}, \mathcal{C})$ of nodal points on a connected irreducible $\mathcal{S}$-*good* real analytic curve $\mathcal{C}$ of a sequence $\mathcal{S}$ of Laplace eigenfunctions φ_j of eigenvalue $-\lambda_j^2$ of a real analytic Riemannian manifold (M, g) of any dimension m is bounded above as follows:

$$n(\varphi_{j_k}, \mathcal{C}) \leq A_{g,\mathcal{C}} \, \lambda_{j_k}.$$

Moreover, we prove that the codimension-two Hausdorff measure $\mathcal{H}^{m-2}(\mathcal{N}_{\varphi_\lambda} \cap H)$ of nodal intersections with a connected, irreducible real analytic hypersurface $H \subset M$ satisfies

$$\mathcal{H}^{m-2}(\mathcal{N}_{\varphi_\lambda} \cap H) \leq A_{g,H} \, \lambda_{j_k}.$$

We further give a geometric control condition on H which is sufficient that H be $\mathcal{S}$-good for a density one subsequence of eigenfunctions.

Remembrance of things past

Theorem

(Toth-Z,'09) Let $\Omega \subset \mathbb{R}^2$ be piecewise analytic and let $n_{\partial\Omega}(\lambda_j)$ be the number of components of the nodal set of the jth Neumann or Dirichlet eigenfunction which intersect $\partial\Omega$. Then, $n_{\partial\Omega}(\lambda_j) \leq C_\Omega \lambda_j$.

Theorem

(Toth-Z '09) Suppose that $\Omega \subset \mathbb{R}^2$ is a C^∞ plane domain, and let $C \subset \Omega$ be a good interior real analytic curve. . Let $n(\lambda_j, C) = \#\mathcal{N}_{\varphi_{\lambda_j}} \cap C$ be the number of intersection points of the nodal set of the j-th Neumann (or Dirichlet) eigenfunction with C. Then there exists $A_{C,\Omega} > 0$ such that $n(\lambda_j, C) \leq A_{C,\Omega} \lambda_j$.

New results

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The first set of new results generalize the plane domain theorems to real anaytic Riemannian manifolds of any dimension. One then must consider what dimension the submanifold C should have. The new results work in all co-dimensions but we only state the results for curves and for hypersurfaces.

The new results also assume $\partial M = \emptyset$. The counting techniques are based on analytic continuation of the wave kernel, which so far have not been generalized to the boundary case.

Results assuming goodness

Theorem

Suppose that (M^m, g) is a real analytic Riemannian manifold of dimension m without boundary and that $\mathcal{C} \subset M$ is connected, irreducible real analytic curve. If $\mathcal{C}$ is $\mathcal{S}$-good, then there exists a constant $A_{\mathcal{S},g}$ so that

$$n(\varphi_j, \mathcal{C}) := \#\{\mathcal{C} \cap \mathcal{N}_{\varphi_j}\} \leq A_{\mathcal{S},g}\, \lambda_j, \quad j \in \mathcal{S}.$$

Theorem

Let (M^m, g) be a real analytic Riemannian manifold of dimension m and let $H \subset M$ be a connected, irreducible, $\mathcal{S}$-good real analytic hyperurface. Then, there exists a constant $C > 0$ depending only on (M, g, H) so that

$$\mathcal{H}^{m-2}(\mathcal{N}_{\varphi_{j_k}} \cap H) \leq C \lambda_{j_k}, \quad (j_k \in \mathcal{S}).$$

Goodness?

Why irreducible? Suppose C_1 is a good curve and C_2 is bad, e.g. the fixed point set of an isometric involution. Then $C_1 \cup C_2$ is good but the counting results do not work.

We now give sufficient geometric control conditions for 'goodness'. The definition of $\mathcal{S}$-good makes sense for any connected, irreducible analytic submanifold $H \subset M$, not only curves.

Notation and assumptions

Given a submanifold $H \subset M$, we denote the restriction operator to H by $\gamma_H f = f|_H$. To simplify notation, we also write $\gamma_H f = f^H$. The criterion that a pair $(H, \mathcal{S})$ be good is stated in terms of the associated sequence

$$u_j := \frac{1}{\lambda_j} \log |\varphi_j|^2 \tag{1}$$

of normalized logarithms, and in particular their restrictions

$$u_j^H := \gamma_H u_j := \frac{1}{\lambda_j} \log |\varphi_j^H|^2 \tag{2}$$

to H. We only consider the goodness of connected, irreducible, real analytic submanifolds.

Definition of Good

Definition: Given a subsequence $\mathcal{S} := \{\varphi_{j_k}\}$*, we say that a connected, irreducible real analytic submanifold* $H \subset M$ *is* $\mathcal{S}$*-good, or that* $(H, \mathcal{S})$ *is a good pair, if the sequence* (2) *with* $j_k \in \mathcal{S}$ *does* **not** *tend to* $-\infty$ *uniformly on compact subsets of* H*, i.e. there exists a constant* $M_{\mathcal{S}} > 0$ *so that*

$$\sup_H u_j^H \geq -M_{\mathcal{S}}, \quad \forall j \in \mathcal{S}.$$

If H *is* $\mathcal{S}$*-good when* $\mathcal{S}$ *is the entire orthonormal basis sequence, we say that* H *is completely good. If* $\mathcal{S}$ *has density one we say that* H *is almost completely good.*

The opposite of a good pair $(H, \mathcal{S})$ is a bad pair.

Equivalent notions of good

The following are equivalent on a real analytic curve.

1. Goodness in the sense of Definition 13, or equivalently in the sense that $\|\varphi_j|_H\|_{L^\infty(H)} \geq e^{-a\lambda_j}$.
2. Goodness in the sense $\|\varphi_j^H\|_{L^2(H)} \geq e^{-a\lambda_j}$.
3. Goodness in the sense that $\frac{1}{\lambda_j} \log|\varphi_j|_H| \to -\infty$ does not hold uniformly on the real H.
4. Goodness in the sense that $\frac{1}{\lambda_j} \log|\varphi_j^{\mathbb{C}}|_H| \to -\infty$ does not hold uniformly on the complex H.

Geometric control conditionsfor Goodness

The criteria consists of two conditions on H:

- (i) asymmetry with respect to geodesic flow, and
- (ii) a full measure flowout condition.

Geodesic asymmetry

The asymmetry condition (i) pertains to the two 'sides' of H, i.e. to the two lifts of $(y, \eta) \in B^* H$ to unit covectors $\xi_{\pm}(y, \eta) \in S^*_H M$ to M. We denote the symplectic volume measure on $B^* H$ by μ_H. We define the symmetric subset $B^*_S H$ to be the set of $(y, \eta) \in B^* H$ so that $G^t(\xi_+(y, \eta)) = G^t(\xi_-(y, \eta))$ for some $t \neq 0$.

Definition: *H is microlocally asymmetric if $\mu_H(B^*_S H) = 0$.* I.e. if we lift an initial tangent vector to H to each side of H, then almost surely the geodesics do not return to the same point at the same time.
This rules out fixed point sets of isometric involutions.

Geometric control

Next we turn to the flow-out condition (ii). It is that

$$\mu_L(\mathrm{FL(H)}) = 1. \tag{3}$$

where

$$\mathrm{FL(H)} := \bigcup_{t \in \mathbb{R}} G^t(S^*_H M \setminus S^* H) \tag{4}$$

is the geodesic flowout of of the non-tangential unit cotangent vectors $S^*_H M \setminus S^* H$ along H. Since H is a hypersurface, $S^*_H M \subset S^* M$ is also a hypersurface which is almost everywhere transverse to the geodesic flow, i.e. it is a symplectic transversal. It follows that the flowout is an invariant set of positive measure in $S^* M$.

Theorem

The next result is a sufficient condition that H be almost completely good.

Theorem

Suppose that H is a microlocally asymmetric hypersurface satisfying $\mu_L(\mathrm{FL(H)}) = 1.$
Then: if $\mathcal{S} = \{\varphi_{j_k}\}$ is a sequence of eigenfunctions satisfying $||\varphi_{j_k}|_H||_{L^2(H)} = o(1)$, then the upper density $D^*(\mathcal{S})$ equals zero.

The following theorem gives a more quantitative version:

Theorem

Let $H \subset M$ be a microlocally asymmetric hypersurface satisfying. $\mu_L(\mathrm{FL(H)}) = 1.$ *Then, for any $\delta > 0$, there exists a subset $\mathcal{S}(\delta) \subset \{1, ..., \lambda\}$ of density $\mathcal{D}(\mathcal{S}(\delta)) \geq 1 - \delta$ such that*

$$\|\varphi_{\lambda_j}\|_{L^2(H)} \geq C(\delta) > 0, \quad j \in \mathcal{S}(\delta).$$

Measures of goodness

There are two natural parameters of $\mathcal{S}$-goodness of H: the density of $\mathcal{S}$ and the rate of decay of eigenfunctions restricted to H.

The geomtric control condition pertains to the decay rate $||\varphi_{j_k}|_H||_{L^2(H)} = o(1)$. Goodness pertains to super-exponential decay. We do not know any general criteria for goodness in the second sense which do not imply goodness in the first.

J. Jung proved that geodesic distance circles and horocycles in the hyperbolic plane are good relative to eigenfunctions on compact or finite area hyperbolic surfaces. L. El-Hajj and J. A. Toth proved that curves of strictly positive geodesic curvature in a convex Euclidean domain are good.

The main result on counting nodal points on curves

Theorem

Let $\mathcal{C}$ be an asymmetric C^ω curve on a compact, closed, C^ω Riemannian surface (M^2, g) satisfying (3). Then, for any $\delta > 0$ there exists a subsequence $\mathcal{S}(\delta)$ with $\mathcal{D}(\mathcal{S}(\delta)) \geq 1 - \delta$ for which $\mathcal{C}$ is $\mathcal{S}'$-good and a constant $A_{\mathcal{S},g}(\delta) > 0$ such that

$$n(\varphi_j, \mathcal{C}) := \#\{\mathcal{C} \cap \mathcal{N}_{\varphi_j}\} \leq A_{\mathcal{S},g}(\delta)\, \lambda_j, \quad j \in \mathcal{S}(\delta).$$

Theorem

Let H be an asymmetric C^ω hypersurface of a compact, closed, C^ω Riemannian manifold (M^m, g) satisfying (3). Then, for any $\delta > 0$ there exists a subsequence $\mathcal{S}(\delta)$ with $\mathcal{D}(\mathcal{S}(\delta)) \geq 1 - \delta$ for which $\mathcal{C}$ is $\mathcal{S}'$-good and a constant $A_{\mathcal{S},g}(\delta) > 0$ such that

$$\mathcal{H}^{m-2}(\mathcal{N}_{\varphi_\lambda} \cap H) \leq A_{\mathcal{S},g}(\delta)\, \lambda_j, \quad j \in \mathcal{S}(\delta).$$

Ideas of proofs

- The upper bound for curves is based on analytic continuation $U(i\tau, Z, y)$ of the wave kernel $U(t, x, y) = \exp it\sqrt{-\Delta}$ to the complexification $M_{\mathbb{C}}$ of M^m (Grauert tube). The analytic continuation of eigenfunctions is given by, $U(i\tau)\varphi_j^{\mathbb{C}} = e^{-\tau\lambda_j}\varphi_j^{\mathbb{C}}$. $U(i\tau, Z, y)$ is an FIO with complex phase of order $\frac{m-1}{4}$, and that gives a growth estimate on $\varphi_j^{\mathbb{C}}$. A Jensen type argument gives upper bounds on zeros on complexified curves.

- For H of higher dimension, one uses Crofton's formula to define $\mathcal{H}^{m-2}(\mathcal{N}_{\varphi_\lambda} \cap H$, then analytically continues and then uses a Jensen type argument.

- The geometric control criterion for goodness comes from studying the relation of microlocal defect measures of eigenfunctions on M and of their restrictions to H.