Bounds on the entanglement entropy of droplet states in the XXZ chain

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Entanglement entropy of quantum spin systems

Hilbert space of spin states on $\Lambda \subset \mathbb{Z}^\nu$**:** $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^d$

Bipartite decomposition $B \subset \Lambda$**:** $\mathcal{H}_\Lambda = \mathcal{H}_B \otimes \mathcal{H}_{B^c}$

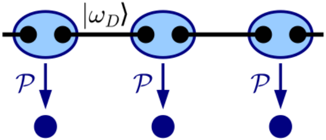
Reduced state on $\mathcal{H}_B$ of **pure state** $\psi \in \mathcal{H}_\Lambda$: $\varrho_B = \mathrm{Tr}_{\mathcal{H}_{B^c}} |\psi\rangle\langle\psi|$

Rényi entropy $\alpha \in [0, \infty)$

$$S_B^{(\alpha)} \equiv S_B^{(\alpha)}(\psi) = \frac{1}{1-\alpha} \ln \mathrm{Tr} \left[(\varrho_B)^\alpha \right] .$$

- $S_B^{(0)} = \ln \mathrm{Rank}\, \varrho_B$ **Hartley** or min-entropy
- $S_B^{(1)} = -\mathrm{Tr}\, (\varrho_B \ln \varrho_B)$ **von Neumann** entropy
- **Monotonicity:** For any $0 \leq \alpha \leq \beta$: $0 \leq S^{(\beta)} \leq S^{(\alpha)}$
- Trivial **volume bound**: $S_B^{(0)} \leq \ln \dim \mathcal{H}_B = |B| \ln d$

1. **Product states:** $\psi = \bigotimes_{x\in\Lambda} \psi_x$, e.g. canonical ONB $\psi_x = e_{j_x}$, $j_x \in \{1, \cdot, d\}$. Then: $S_B^{(0)}(\psi) = 0\,.$



2. **MPS / PEPS:**

   Entangled pair for each bond: $\omega_D = \frac{1}{\sqrt{D}} \sum_{j=1}^{D} e_j \otimes e_j \in \mathbb{C}^D \otimes \mathbb{C}^D$

   'Projection' onto physical space: $\mathcal{P} : \mathbb{C}^D \otimes \mathbb{C}^D \to \mathbb{C}^d$

$$\langle e_j\,,\, \mathcal{P}\; e_\xi \otimes e_\eta \rangle = A^{(j)}_{\xi.\eta}$$

$$\psi = \sum_{j_1,\dots,j_{|\Lambda|}} \mathrm{Tr}\left(A^{(1)} \cdots A^{(|\Lambda|)}\right)\; e_{j_1} \otimes \cdots \otimes e_{j_{|\Lambda|}}$$

   Then: $S_B^{(0)}(\psi) \leq |\partial B|\; \ln D\,.$ **area law**

Known results

1. The average entropy of **random pure states** on $\mathcal{H}_\Lambda$ (i.e. Haar measure on the unit sphere) exhibits a **volume law**.
   Lubkin '78, Page '93
2. The entanglement entropy of the **ground-state** of any **gapped, one-dimensional spin systems** obeys an **area law**.
   Hastings '07
   Aharonov/Arad/Vazirani/Landau '11
3. For one-dimensional spin systems **exponential decay of correlatons** implies an **area law**. Brandao/Horodecki '12
4. The ground-state of **free** (lattice) **fermions** obeys an area law with a **logarithmic correction**:

$$S_B^{(\alpha)} = c_\alpha \, |\partial B| \, \ln |B| + o(|\partial B| \, \ln |B|)$$

and the same applies to the ground state of the XY spin chain.

Vidal/Latorre/Rico/Kitaev '03
Jin/Korepin '04, Wolf '05
...
Gioev/Klich '06, Helling/Leschke/Spitzer/Sobolev $\geq$ '11

The XXZ spin chain and its Ising phase

XXZ spin-$\frac{1}{2}$ chain on a finite interval $\Lambda = [-L, L] \cap \mathbb{Z}$

$$H = -\frac{1}{2}\sum_{x=-L}^{L-1}\left(\sigma_x^1\sigma_{x+1}^1 + \sigma_x^2\sigma_{x+1}^2\right) - \frac{\Delta}{2}\sum_{x=-L}^{L-1}\left(\sigma_x^3\sigma_{x+1}^3 - 1\right) + \frac{\Delta}{2}\left(2 - \sigma^3_{-L} - \sigma^3_{-L}\right)$$

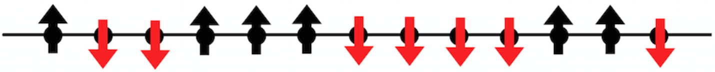
- $\Delta = 0$ XX spin chain
- $\Delta = 1$ Heisenberg ferromagnet
- $\Delta > 1$ **Ising phase**

- In case $\Delta > 0$ the **boundary term** discourages down spins at the boundary.

Equivalence of XXZ to interacting particle system

Particle interpretation of eigenbasis of σ_x^3, $x \in \Lambda_L$:

$L = 6$ $n = 7$



Configuration space of n hard-core particles on $\Lambda := [-L, L] \cap \mathbb{Z}$:

$$\mathcal{X}_\Lambda^n := \left\{ \mathbf{x} = \{x_1, x_2, \dots, x_n\} \in \Lambda_L^n \,\middle|\, x_1 < x_2 < \dots < x_n \right\}$$

Unitary equivalence: $\mathcal{V} : \mathcal{H}_\Lambda \to \bigoplus_{n=0}^{2L+1} \ell^2\left(\mathcal{X}_\Lambda^n\right)$, $\quad \mathcal{V} H \mathcal{V}^* = -A + 2\Delta\, W$

- Adjacency operator on $\mathcal{X}_\Lambda^n$: $\quad (A\psi)(\mathbf{x}) = \sum_{\substack{\mathbf{y} \in \mathcal{X}_\Lambda^n \\ d(\mathbf{x},\mathbf{y})=1}} \psi(\mathbf{y})$

  $$d(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{n} |x_j - y_j|$$

- Interaction potential $\quad W(\mathbf{x}) = \frac{1}{2}\, \#\{\text{cluster bounderies in } \mathbf{x}\}$

Conservation law: $\left[H_{\mathrm{XXZ}}, \sum_x \sigma_x^3\right] = 0$ implies block structure of H!

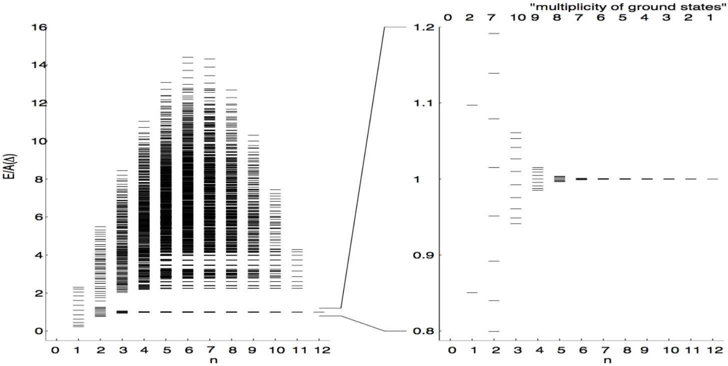
Ground state at $n = 0$: $\psi_0 = \bigotimes_{x\in\Lambda} |\uparrow\rangle, \qquad H\psi_0 = 0.$

In case $n \geq 1$ a **cluster decomposition** $\mathcal{X}_\Lambda^n = \bigcup_{k=1}^{n} \mathcal{C}_k^n$
energetically distinguishes states:

- Interaction for k-cluster configurations $\mathbf{x} \in \mathcal{C}_k^n$: $\qquad W\delta_\mathbf{x} := k\delta_\mathbf{x}$.
- Orthogonal projection Q_k^n onto the subspace $\bigoplus_{j=k}^{\infty} \ell^2(\mathcal{C}_j^n)$ of **at least k clusters**:

$$Q_k^n H \, Q_k^n \geqslant 2k(\Delta - 1)$$

The spectrum in the infinite volume limit is explicit through **Bethe Ansatz**.
In finite-volume with BC:



Droplet band for fixed n corresponding to one cluster $\mathcal{C}^n \equiv \mathcal{C}_1^n$:

$$\Delta^n := 2\sqrt{\Delta^2 - 1}\left[\frac{\cosh(\rho_\Delta n) - 1}{\sinh(\rho_\Delta n)}, \frac{\cosh(\rho_\Delta n) - 1}{\sinh(\rho_\Delta n)}\right] \subset [2(\Delta - 1), 2(\Delta + 1)]$$

where $\rho_\Delta := \ln(\Delta + \sqrt{\Delta^2 - 1})$

Theorem

For $\Delta > 1$, $\mu_\mathrm{T} > 0$ and

$$0 \le E < 4\Delta - 12\, e^{\mu_\mathrm{T}} ,$$

the Green function $G_\Lambda^{(2)}(\mathbf{x},\mathbf{y};E) := \langle \delta_\mathbf{x}, \left(Q_2 H_\Lambda Q_2 - E\right)^{-1} \delta_\mathbf{y} \rangle$ of the projection Q_2 on all non-clustered configurations satisfies:

$$\boxed{|G_\Lambda^{(2)}(\mathbf{x},\mathbf{y};E)| \leqslant C_\mathrm{T} e^{-\mu_\mathrm{T} d(\mathbf{x},\mathbf{y})}}$$

for all $n \ge 2$ all $\mathbf{x},\mathbf{y} \notin \mathcal{C}^n$ with $d(\mathbf{x},\mathbf{y}) = \sum_j |x_j - y_j|$ at some $C_\mathrm{T} < \infty$.

- A proof combines the low entropy of available neighbors for droplet states with the energetic penalty of cluster break-up

  Beaud/W. '17, Elgart/Klein/Stolz '17

- Consequence for the spectral projection onto droplet states:

  For any $I \subset \mathbb{R}$ with $\sup I < 4\Delta - 12$ there is $C, \mu \in (0,\infty)$ such that for all $n \ge 1$, all Λ and all $\mathbf{x} \in \mathcal{X}_\Lambda^n$:

$$N_I^n(\mathbf{x}) := \langle \delta_\mathbf{x}, P_I(H)\delta_\mathbf{x} \rangle \le C\, e^{-\mu d(\mathbf{x},\mathcal{C}^n)} .$$

Main result

Theorem (Beaud/W. '17)

Let ψ be a state with $n \geq 1$ particles in the droplet spectrum, i.e. $\psi = P_I(H)\psi$ with

$$\sup I < 4\Delta - 12 \, .$$

Then for any $\alpha \in (0,1)$ there are constants $c_\alpha, C_\alpha \in (0, \infty)$ (which are independent of ψ and n) such that

$$\boxed{S_B^{(\alpha)}(\psi) \leq c_\alpha \ln \min\{n, |B|\} + C_\alpha}$$

for any non-vanishing interval $B \subset \Lambda$ and all Λ.

- Extension to ψ not necessarily of fixed particle number.
- Applies to **quantum quench**, i.e. $\psi_t = e^{-itH}\psi$ with ψ as above.

The simple proof idea

$$
\begin{aligned}
\mathrm{Tr}\left[(\varrho_B)^\alpha\right] &\leq \sum_{m=0}^{\min\{n,|B|\}} \sum_{\mathbf{x}\in\mathcal{X}_\Lambda^m} \langle \delta_{\mathbf{x}}, (\varrho_B)^\alpha\, \delta_{\mathbf{x}} \rangle \\
&\leq 2 + \sum_{m=1}^{\min\{n,|B|-1\}} \sum_{\mathbf{x}\in\mathcal{X}_\Lambda^m} \langle \delta_{\mathbf{x}}, \varrho_B\, \delta_{\mathbf{x}} \rangle^\alpha \\
&\leq 2 + \sum_{m=1}^{\min\{n,|B|-1\}} \sum_{\substack{\mathbf{x}\cap B\neq\emptyset \\ \mathbf{x}\cap B^c\neq\emptyset}} N_l^m(\mathbf{x})^\alpha \leq\; 2 + C_\alpha \sum_{m=1}^{\min\{n,|B|-1\}} \sum_{\substack{\mathbf{x}\cap B\neq\emptyset \\ \mathbf{x}\cap B^c\neq\emptyset}} e^{-\mu_\alpha d(\mathbf{x},\mathcal{C}^m)} \\
&\leq\; c_\alpha + C_\alpha\; \min\{n, |B|\}
\end{aligned}
$$

One relevant observation:

$$
\sup_{\mathbf{v}\in\mathcal{C}^n} \sum_{\mathbf{x}\in\mathcal{X}_{\mathbb{Z}}^n} e^{-\mu d(\mathbf{x},\mathbf{v})} \leqslant \frac{1}{1-e^{-\mu}} \left(\prod_{k=1}^{\infty} \frac{1}{1-e^{-k\mu}} \right)^2 < \infty\,.
$$

A note on the case with a non-negative (random) magnetic field

Adding a **non-negative, random magnetic field** in the 3 direction:

$$H_\omega = H + \sum_x \omega_x \left(1 - \sigma_x^3\right)$$

leaves the Combes-Thomas bound unchanged.

Random term on the n particle subspace is typically: $\mathcal{O}(n)$

Lemma (cf. Aizenman/W. '09)

Let $I = [0, \sup I]$ and $\lambda > 0$. There are $C, c \in (0,\infty)$ s.t. for all n, L and $\mathbf{x} \in \mathcal{X}_\Lambda^n$:

$$\mathbb{E}\left[N_I^n(\mathbf{x})\right] \leqslant Ce^{-cn}.$$

Corollary:

$$\mathbb{E}\left[\sup_\psi \sup_{t\in\mathbb{R}} \exp\left\{(1-\alpha)\, S_U^{(\alpha)}\left[e^{-itH}\psi\right]\right\}\right] \le C$$

where the supremum is taken over all normalized states in the droplet regime, i.e. $\psi = P_I(H)\psi$ with $\sup I < 4\Delta - 12$.

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- Similar result for the XY spain chain:
  Abduhl-Rahman/Nachtergaele/Sims/Stolz '16

Thank you for organizing this nice event

in such a splendid location!