Non-equilibrium almost-stationary states for interacting electrons on a lattice

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Based on joint work with **Domenico Monaco**.

1. Example and setup

As a microscopic model for a quantum Hall system consider a system of interacting fermions on the domain Λ, where $\Lambda \subset \mathbb{Z}^2$ is the centred square of side-length L with the vertical edges identified.

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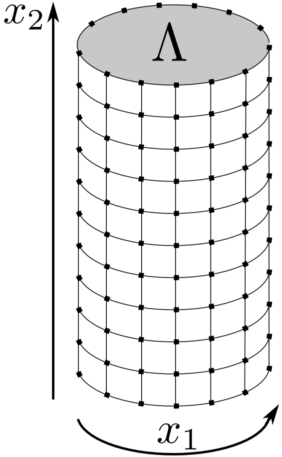

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A typical Hamiltonian could be of the form

$$\begin{aligned} H_0^\Lambda &= \sum_{(x,y)\in\Lambda^2} a_x^* \, T(x \overset{\Lambda}{-} y) \, a_y + \sum_{x\in\Lambda} a_x^* \phi(x) a_x \\ &+ \sum_{\{x,y\}\subset\Lambda} a_x^* a_x \, W(d^\Lambda(x,y)) \, a_y^* a_y - \mu \, \mathfrak{N}_\Lambda \,, \end{aligned}$$

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In the following by a "local Hamiltonian" we mean a family $A = \{A^\Lambda\}_\Lambda$ of self-adjoint operators A^Λ indexed by the system size Λ and possibly other parameters that is a "sum of local terms". Typically

$$\|A^\Lambda\| \sim |\Lambda| = L^d \,.$$

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Assume that $H_0 = \{H_0^\Lambda\}$ has a ground state that is gapped uniformly in the system size $|\Lambda|$, i.e.

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Note that the potential difference of εL at the two edges is, for sufficiently large system size L, larger than the spectral gap g. Thus, the perturbed Hamiltonian

$$H^{\varepsilon,\Lambda} := H_0^\Lambda + V^{\varepsilon,\Lambda}$$

no longer has a meaningful gap above the ground state.

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This state is not the ground state of $H^{\varepsilon,\Lambda}$, nor is it any other equilibrium state of $H^{\varepsilon,\Lambda}$, since, for example, the local Fermi-levels at the opposite edges are expected to be different.

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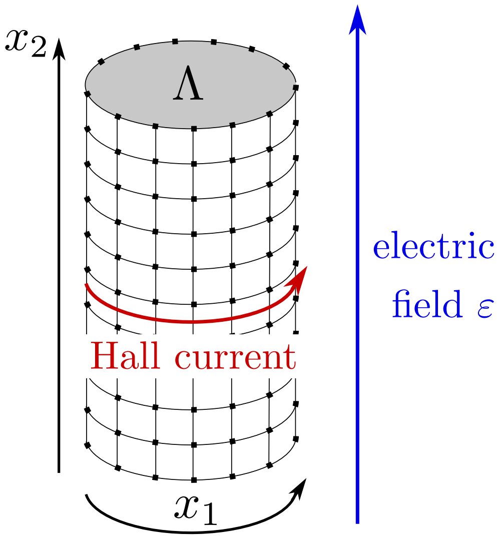

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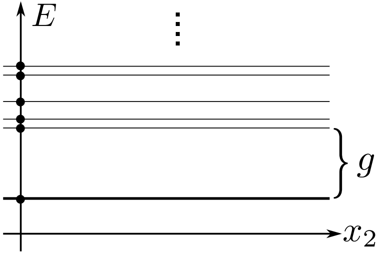
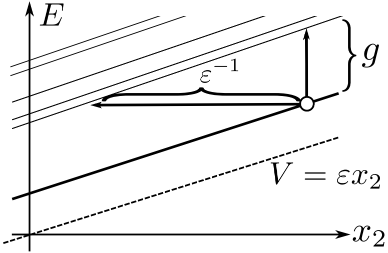
Heuristic picture suggesting the existence of a non-equilibrium almost-stationary state (NEASS):





2. Results

Let H_0 and H_1 be families of self-adjoint local Hamiltonians, let H_0 have a gapped ground state, let V_v be a slowly varying potential, and put

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Theorem (Non-equilibrium almost-stationary states)

There is a sequence of self-adjoint local Hamiltonians S_n, such that for any $n \in \mathbb{N}$ the projector

$$\Pi_n^{\varepsilon,\Lambda} := \mathrm{e}^{\mathrm{i}\varepsilon S_n^{\varepsilon,\Lambda}}\, P_0^\Lambda\, \mathrm{e}^{-\mathrm{i}\varepsilon S_n^{\varepsilon,\Lambda}}$$

satisfies

$$[\Pi_n^{\varepsilon,\Lambda}, H^{\varepsilon,\Lambda}] = \varepsilon^{n+1}\, [\Pi_n^{\varepsilon,\Lambda}, R_n^{\varepsilon,\Lambda}]$$

for some local R_n.

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for some local R_n.

Let $\rho^{\varepsilon,\Lambda}(t)$ be the solution of the Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\rho^{\varepsilon,\Lambda}(t) = [H^{\varepsilon,\Lambda}, \rho^{\varepsilon,\Lambda}(t)] \qquad \text{with} \qquad \rho^{\varepsilon,\Lambda}(0) = \Pi_n^{\varepsilon,\Lambda}\,.$$

Then there is a constant C independent of Λ such that for any local Hamiltonian B it holds that

$$\sup_\Lambda \tfrac{1}{|\Lambda|} \left| \mathrm{tr}\left(\rho^{\varepsilon,\Lambda}(t) B^\Lambda\right) - \mathrm{tr}\left(\Pi_n^{\varepsilon,\Lambda} B^\Lambda\right) \right| \leq C\,\varepsilon^{n+1}\, |t|(1+|t|^d)\, |||B|||\,.$$

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Let $f : \mathbb{R} \to [0,1]$ be a smooth "switching" function, i.e. $f(t) = 0$ for $t \leq 0$ and $f(t) = 1$ for $t \geq T > 0$, and define

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Theorem (Adiabatic switching)

The solution of the adiabatic time-dependent Schrödinger equation

$$\mathrm{i}\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\rho^{\varepsilon,\Lambda}(t) = [H^{\varepsilon,\Lambda}(t), \rho^{\varepsilon,\Lambda}(t)] \qquad \text{with} \qquad \rho^{\varepsilon,\Lambda}(0) = P_0^{\Lambda}$$

satisfies for all $t \geq T$ that for any $n \in \mathbb{N}$ there exists a constant C such that for any local Hamiltonian B

$$\sup_{\Lambda} \tfrac{1}{|\Lambda|} \left| \mathrm{tr}\left(\rho^{\varepsilon,\Lambda}(t) B^{\Lambda}\right) - \mathrm{tr}\left(\Pi_n^{\varepsilon,\Lambda} B^{\Lambda}\right)\right| \leq C\, \varepsilon^{n-d}\, |t|(1+|t|^d)\, |||B|||\,.$$

3. Example continued

In the quantum Hall example from the beginning take the current operator

$$J_1^\Lambda = \partial_{\alpha_1} H_0^\Lambda(\alpha)|_{\alpha=0}$$

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$$j_{\mathrm{Hall},1}^\Lambda \quad = \quad \frac{1}{|\Lambda|}\left(\mathrm{tr}(\Pi_n^{\varepsilon,\Lambda} J_1^\Lambda) - \mathrm{tr}(P_0^\Lambda J_1^\Lambda)\right) + \mathcal{O}(\varepsilon^{n-2})$$

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$$\begin{aligned} j^\Lambda_{\mathrm{Hall},1} &= \frac{1}{|\Lambda|}\Big(\mathrm{tr}(\Pi_n^{\varepsilon,\Lambda} J_1^\Lambda) - \mathrm{tr}(P_0^\Lambda J_1^\Lambda)\Big) + \mathcal{O}(\varepsilon^{n-2}) \\ &= \frac{\varepsilon}{|\Lambda|}\,\mathrm{tr}(P_1^{\varepsilon,\Lambda} J_1^\Lambda) + \mathcal{O}(\varepsilon^2)\,, \end{aligned}$$

uniformly in the system size.

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Inserting the explicit expression for $P_1^{\varepsilon,\Lambda}$, we obtain for the Hall conductivity Kubo's "current-current-correlation" formula

$$\sigma_{\mathrm{Hall}}^\Lambda := \frac{j_{\mathrm{Hall},1}^\Lambda}{\varepsilon} = \frac{\mathrm{i}}{|\Lambda|}\,\mathrm{tr}\left(P_0^\Lambda\,\left[\partial_{\alpha_1} P_0^\Lambda(\alpha)|_{\alpha=0}, \left[X_2, P_0^\Lambda\right]\right]\right) + \mathcal{O}(\varepsilon)\,.$$

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- If the perturbation and/or the observable are localized, the result holds with the corresponding normalisation of the trace.

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- We actually prove a general **space-time adiabatic theorem**, similar to what we called **space-adiabatic perturbation theory** long ago (**Panati, Spohn, T. (2003)**).
- The new proof in the context of interacting systems and error bounds uniform in the system size is inspired by the recent adiabatic theorem of **Bachmann, de Roeck, Fraas (2017)**.
- The most important technical ingredient is the local inverse of the Liouvillian that was constructed in the context of the quasi-adiabatic evolution (aka spectral flow) based on Lieb-Robinson bounds. (**Hastings et al. (2005), Nachtergaele et al. (2012)**)

5. References

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