Non-equilibrium almost-stationary states for interacting electrons on a lattice

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Based on joint work with Domenico Monaco.
1. Example and setup

As a microscopic model for a quantum Hall system consider a system of interacting fermions on the domain $\Lambda$, where $\Lambda \subset \mathbb{Z}^2$ is the centred square of side-length $L$ with the vertical edges identified.
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A typical Hamiltonian could be of the form

$$H^\Lambda_0 = \sum_{(x,y) \in \Lambda^2} a^*_x T(x^\Lambda - y) a_y + \sum_{x \in \Lambda} a^*_x \phi(x) a_x$$

$$+ \sum_{\{x,y\} \subset \Lambda} a^*_x a_y \mathcal{W}(d^\Lambda (x, y)) a^*_y a_y - \mu \mathcal{N}_\Lambda ,$$

where $a^*_{x,i}$ and $a_{x,i}$ are standard fermionic creation and annihilation operators at the sites $x \in \Lambda$. 
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A typical Hamiltonian could be of the form

$$H_0^\Lambda = \sum_{(x,y) \in \Lambda^2} a_x^* T(\Lambda^x - y) a_y + \sum_{x \in \Lambda} a_x^* \phi(x) a_x$$

$$+ \sum_{\{x,y\} \subset \Lambda} a_x^* a_x \mathcal{W}(d^\Lambda(x,y)) a_y^* a_y - \mu \mathcal{N}_\Lambda,$$

where $a_{x,i}^*$ and $a_{x,i}$ are standard fermionic creation and annihilation operators at the sites $x \in \Lambda$.

In the following by a “local Hamiltonian” we mean a family $A = \{A^\Lambda\}_\Lambda$ of self-adjoint operators $A^\Lambda$ indexed by the system size $\Lambda$ and possibly other parameters that is a “sum of local terms”. Typically

$$\|A^\Lambda\| \sim |\Lambda| = L^d.$$
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Assume that $H_0 = \{H_0^\Lambda\}$ has a ground state that is gapped uniformly in the system size $|\Lambda|$, i.e.

$$\inf_{\Lambda} \text{dist} \left( E_0^\Lambda, \sigma(H_0^\Lambda) \setminus \{E_0^\Lambda\} \right) = g > 0.$$
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Now add the potential of an electric field of magnitude \( \varepsilon \) pointing in the 2-direction,

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Now add the potential of an electric field of magnitude $\varepsilon$ pointing in the 2-direction,

$$V^{\varepsilon,\Lambda} := \sum_{x \in \Lambda} \varepsilon x_2 a_x^* a_x.$$ 

Note that the potential difference of $\varepsilon L$ at the two edges is, for sufficiently large system size $L$, larger than the spectral gap $g$. Thus, the perturbed Hamiltonian

$$H^{\varepsilon,\Lambda} := H_0^\Lambda + V^{\varepsilon,\Lambda}$$

no longer has a meaningful gap above the ground state.
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Once the field has reached its final value, one expects that the system is in a (almost) stationary state that, in particular, could carry a stationary, non-vanishing Hall current along the closed direction of the cylinder.
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This state is not the ground state of $H^{\varepsilon,\Lambda}$, nor is it any other equilibrium state of $H^{\varepsilon,\Lambda}$, since, for example, the local Fermi-levels at the opposite edges are expected to be different.
1. Example and setup

$x_2$ 

Hall current

$x_1$

electric field $\varepsilon$
1. Example and setup

Heuristic picture suggesting the existence of a non-equilibrium almost-stationary state (NEASS):
2. Results

Let $H_0$ and $H_1$ be families of self-adjoint local Hamiltonians, let $H_0$ have a gapped ground state, let $V_\nu$ be a slowly varying potential, and put

$$H := H_0 + V_\nu + \epsilon H_1.$$
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**Theorem (Non-equilibrium almost-stationary states)**

There is a sequence of self-adjoint local Hamiltonians $S_n$, such that for any $n \in \mathbb{N}$ the projector

$$\Pi_{\epsilon, \Lambda} := e^{i\epsilon S_{\epsilon, \Lambda}} P_0 e^{-i\epsilon S_{\epsilon, \Lambda}}$$

satisfies

$$[\Pi_{\epsilon, \Lambda}, H_{\epsilon, \Lambda}] = \epsilon^{n+1} [\Pi_{\epsilon, \Lambda}, R_{\epsilon, \Lambda}]$$

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for some local $R_n$.

Let $\rho_{\varepsilon, \Lambda}(t)$ be the solution of the Schrödinger equation

$$i \frac{d}{dt} \rho_{\varepsilon, \Lambda}(t) = [H_{\varepsilon, \Lambda}, \rho_{\varepsilon, \Lambda}(t)] \quad \text{with} \quad \rho_{\varepsilon, \Lambda}(0) = \Pi_{\varepsilon, \Lambda}^{\varepsilon, \Lambda}.$$

Then there is a constant $C$ independent of $\Lambda$ such that for any local Hamiltonian $B$ it holds that

$$\sup_{\Lambda} \frac{1}{|\Lambda|} \left| \text{tr} \left( \rho_{\varepsilon, \Lambda}(t) B^\Lambda \right) - \text{tr} \left( \Pi_{\varepsilon, \Lambda}^{\varepsilon, \Lambda} B^\Lambda \right) \right| \leq C \varepsilon^{n+1} |t|(1 + |t|^d) \|B\|.$$
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Let \( f : \mathbb{R} \rightarrow [0, 1] \) be a smooth “switching” function, i.e. \( f(t) = 0 \) for \( t \leq 0 \) and \( f(t) = 1 \) for \( t \geq T > 0 \), and define

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H(t) := H_0 + f(t)(V_v + \varepsilon H_1).
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\]

**Theorem (Adiabatic switching)**

The solution of the adiabatic time-dependent Schrödinger equation

\[
i\varepsilon \frac{d}{dt} \rho^{\varepsilon, \Lambda}(t) = [H^{\varepsilon, \Lambda}(t), \rho^{\varepsilon, \Lambda}(t)] \quad \text{with} \quad \rho^{\varepsilon, \Lambda}(0) = P_0^\Lambda
\]

satisfies for all \( t \geq T \) that for any \( n \in \mathbb{N} \) there exists a constant \( C \) such that for any local Hamiltonian \( B \)

\[
\sup_{\Lambda} \frac{1}{|\Lambda|} \left| \mathrm{tr} \left( \rho^{\varepsilon, \Lambda}(t) B^\Lambda \right) - \mathrm{tr} \left( \Pi_n^{\varepsilon, \Lambda} B^\Lambda \right) \right| \leq C \varepsilon^{n-d} |t|(1 + |t|^d) \|B\|.
\]
3. Example continued

In the quantum Hall example from the beginning take the current operator
\[ J_1^\Lambda = \partial_{\alpha_1} H_0^\Lambda(\alpha)|_{\alpha=0} \]
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as the observable. Then the Hall current density satisfies

\[ j_{\text{Hall},1}^\Lambda \equiv \frac{1}{|\Lambda|} \left( \text{tr} (\prod_n^{\xi,\Lambda} J_1^\Lambda) - \text{tr} (P_0^\Lambda J_1^\Lambda) \right) + O(\varepsilon^{n-2}) \]
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as the observable. Then the Hall current density satisfies

\[ j_{\text{Hall}}^{\Lambda,1} = \frac{1}{|\Lambda|} \left( \text{tr}(\prod_n P_{n}^{\Lambda,1} J_1^{\Lambda}) - \text{tr}(P_0^{\Lambda} J_1^{\Lambda}) \right) + O(\varepsilon^{n-2}) \]

\[ = \frac{\varepsilon}{|\Lambda|} \text{tr}(P_1^{\Lambda,1} J_1^{\Lambda}) + O(\varepsilon^2), \]

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as the observable. Then the Hall current density satisfies

\[
 j_{\text{Hall},1}^\Lambda = \frac{1}{|\Lambda|} \left( \text{tr}(\Pi_n^\varepsilon,^\Lambda J_1^\Lambda) - \text{tr}(P_0^\Lambda J_1^\Lambda) \right) + O(\varepsilon^{n-2})
\]

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 = \frac{\varepsilon}{|\Lambda|} \text{tr}(P_1^{\varepsilon,^\Lambda} J_1^\Lambda) + O(\varepsilon^2),
\]

uniformly in the system size.

Inserting the explicit expression for \( P_1^{\varepsilon,^\Lambda} \), we obtain for the Hall conductivity Kubo’s “current-current-correlation” formula

\[
 \sigma_{\text{Hall}}^\Lambda := \frac{j_{\text{Hall},1}^\Lambda}{\varepsilon} = \frac{1}{|\Lambda|} \text{tr} \left( P_0^\Lambda \left[ \partial_{\alpha_1} P_0^\Lambda (\alpha)|_{\alpha=0}, \left[ X_2, P_0^\Lambda \right] \right] \right) + O(\varepsilon).
\]
4. Remarks

- If the perturbation and/or the observable are localized, the result holds with the corresponding normalisation of the trace.
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- We actually prove a general **space-time adiabatic theorem**, similar to what we called **space-adiabatic perturbation theory** long ago ([Panati, Spohn, T. (2003)]).
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- The new proof in the context of interacting systems and error bounds uniform in the system size is inspired by the recent adiabatic theorem of Bachmann, de Roeck, Fraas (2017).
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- We actually prove a general space-time adiabatic theorem, similar to what we called space-adiabatic perturbation theory long ago (Panati, Spohn, T. (2003)).

- The new proof in the context of interacting systems and error bounds uniform in the system size is inspired by the recent adiabatic theorem of Bachmann, de Roeck, Fraas (2017).

- The most important technical ingredient is the local inverse of the Liouvillian that was constructed in the context of the quasi-adiabatic evolution (aka spectral flow) based on Lieb-Robinson bounds. (Hastings et al. (2005), Nachtergaele et al. (2012))
5. References


Thanks for your attention!
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