The Bogolubov-de Gennes Equations

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based on the joint work with Li Chen

previous work with V. Bach, S. Breteaux, Th. Chen
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Discussions with Rupert Frank and Christian Hainzl

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Hartree and Hartree-Fock Equations

Starting with the many-body Schrödinger equation

\[ i\partial_t \psi = H_n \psi, \]

for a system of \( n \) identical bosons or fermions and restricting it to the Hartree and Hartree-Fock states

\[ \otimes_1^n \psi \quad \text{and} \quad \wedge_1^n \psi, \]

we obtain the Hartree and the Hartree-Fock equations.

There is a considerable literature on

- the derivation of the Hartree and Hartree-Fock equations
- the existence theory
- the ground state theory
- the excitation spectrum

Describing quantum fluids, like superfluids and superconductors, requires another conceptual step.
Non-Abelian random Gaussian fields

We think of Hartree-Fock states as non-Abelian generalization of random Gaussian fields. These fields (centralized) are uniquely characterized by the expectations of the 2nd order:

$$\langle \psi^*(y) \psi(x) \rangle.$$  \hspace{1cm} (1)

We generalize this to (centralized) quantum fields, $\hat{\psi}(x)$, by assuming that the latter are uniquely characterized by the expectations of the 2nd order:

$$\langle \hat{\psi}^*(y) \hat{\psi}(x) \rangle.$$  \hspace{1cm} (2)

These are exactly the Hartree-Fock states.

However, the above states are not the most general ‘quadratic’ states. The most general ones are defined by

$$\langle \hat{\psi}^*(y) \hat{\psi}(x) \rangle \text{ and } \langle \hat{\psi}(x) \hat{\psi}(y) \rangle.$$  \hspace{1cm} (3)
Quantum fluids

To sum up, the most general ‘quantum Gaussian’ states are the states defined by their quadratic expectations

\[ \gamma(x, y) := \langle \hat{\psi}^*(y) \hat{\psi}(x) \rangle, \]  
\[ \alpha(x, y) := \langle \hat{\psi}(x) \hat{\psi}(y) \rangle. \]

(4)  
(5)

\(\alpha\) describes the (macroscopic) pair coherence (correlation amplitude, long-range order).

This type of states were introduced by Bardeen-Cooper-Schrieffer and further elaborated by Bogolubov.

In math language, these are the quasifree states. They give the most general one-body approximation to the \(n\)–body dynamics.

Let \(\gamma\) and \(\alpha\) denote the operators with the integral kernels \(\gamma(x, y)\) and \(\alpha(x, y)\). Then, after stripping off the spin components,

\[ \gamma = \gamma^* \geq 0 \quad \text{and} \quad \alpha^* = \bar{\alpha} \quad \text{and a technical property,} \]

(6)

where \(\bar{\sigma} = C\sigma C\) with \(C\) the complex conjugation.
Quasifree reduction

Following V. Bach, S. Breteaux, Th. Chen and J. Fröhlich and IMS, we map the solution \( \omega_t \) of the Schrödinger equation

\[
i \partial_t \omega_t(A) = \omega_t([A, H]), \quad \forall A \in \mathcal{M}.
\]  

(7)

to the family \( \varphi_t \) of quasifree states satisfying

\[
i \partial_t \varphi_t(A) = \varphi_t([A, H]) \quad \forall \text{ quadratic } A.
\]  

(8)

We call this map the quasifree reduction.

Evaluating (8) on \( \hat{\psi}^*(x, t)\hat{\psi}(y, t) \), \( \hat{\psi}(x, t)\hat{\psi}(y, t) \), yields a system of coupled nonlinear PDE’s for \( (\gamma_t, \alpha_t) \).

For the standard any-body hamiltonian, these give the (time-dependent) Bogolubov-de Gennes (fermions) or Hartree-Fock-Bogolubov (bosons) equations.

(In the latter case, one has also \( \phi_t(x) = \langle \hat{\psi}(y, t) \rangle \).)

The BdG eqs give an equivalent formulation of the BCS theory.
Dynamics (Bosons)

Derivation (formal) and analysis of the dynamics for the generalized Gaussian states for bosons:

V. Bach, S. Breteaux, Th. Chen and J. Fröhlich and IMS. (See Grillakis and Machedon for some rigorous results on the deriv.)

For the pair interaction potential \( \nu = \lambda \delta \) (where \( \lambda \in \mathbb{R} \) and \( \delta \) is the delta distribution), they are of the form,\(^1\)

\[
\begin{align*}
  i \partial_t \phi_t &= h\phi_t + \lambda |\phi_t|^2 \phi_t + 2\lambda \rho_{\gamma t} \phi_t + \lambda \overline{\phi}_t \rho_{\alpha t} \\
  i \partial_t \gamma_t &= [h_{\gamma t, \alpha t}, \gamma_t]_- + \lambda [w_t, \alpha_t]_- , \\
  i \partial_t \alpha_t &= [h_{\gamma t, \alpha t}, \alpha_t]_+ + \lambda [w_t, \gamma_t]_+ + \lambda w_t ,
\end{align*}
\]

where \( h \) is a one-particle Schrödinger operator, \( \rho_{\mu}(x) := \mu(x; x) \),

\[
\begin{align*}
  w_t(x) &= \rho_{\alpha t}(x) + \phi_t^2(x) , \\
  h_{\gamma t, \alpha t} &= h + 2\lambda (|\phi_t|^2 + \rho_{\gamma t}) .
\end{align*}
\]

\(^1\)[\(A, B\)_- = \(AB^* - BA^*\) and \([A, B]_+ = AB^T + BA^T\), with \(A^T = \overline{A}^*\).]
Dynamics (Fermions)

From now on, we concentrate on fermions.

It is convenient to organize the operators $\gamma$ and $\alpha$ into the matrix-operator

$$\eta := \begin{pmatrix} \gamma & \alpha \\ \bar{\alpha} & 1 \pm \bar{\gamma} \end{pmatrix}$$

(13)

Then

$$0 \leq \gamma = \gamma^* \leq 1 \quad \text{and} \quad \alpha^* = \bar{\alpha}$$

and a technical property

$$\iff \quad 0 \leq \eta = \eta^* \leq 1$$

(14)

As the generalized Gaussian states for fermions describe superconductors we have to couple the order parameter $\eta$ to the electromagnetic field.

We describe the latter by the magnetic and electric potentials, $a$ and $\phi$.

Then states of the fermionic system are now described by the triple $(\eta, a, \phi)$, where $\eta \sim (\gamma, \alpha)$. 
Bogolubov-de Gennes Equations

The many-body Schrödinger equation implies the equations

\[
i(\partial_t + i\phi)\eta = [H(\eta, a), \eta], \tag{15}\]
\[
\partial_t(\partial_t a + \nabla\phi) = -\text{curl}^* \text{curl} a + j(\gamma, a), \tag{16}\]

where \(j(\gamma, a)(x) := [-i\nabla a, \gamma]_+(x, x)\), the superconducting current,

\[
H(\eta, a) = \left( \frac{h_\gamma a}{v^#_\alpha - h_\gamma a} \right),
\]

where \(v^# : \alpha(x, y) \rightarrow v(x, y)\alpha(x; y)\), \(v(x, y)\) is a pair potential, and

\[
h_\gamma a = -\Delta_a + v^*\gamma, \tag{17}\]

with \(\Delta_a := (\nabla + ia)^2\) and \(v^*\gamma := v * \rho_\gamma, \rho_\gamma(x) := \gamma(x, x)\), the direct self-interaction energy. (We dropped the exchange energy.)

These are the celebrated Bogolubov-de Gennes equations (BdG eqs). They give an equivalent description of the BCS theory.
Conservation laws

BdG eqs conserve the energy $E(\eta, a, e) := E(\eta, a) + \frac{1}{2} \int |e|^2$, where

$$E(\eta, a) = \text{Tr} \left( (-\Delta_a)\gamma \right) + \frac{1}{2} \text{Tr} \left( (v * \rho_\gamma)\gamma \right)$$

$$+ \frac{1}{2} \text{Tr} \left( \alpha^* v^#\alpha \right) + \frac{1}{2} \int dx |\text{curl} \ a(x)|^2$$  \hspace{1cm} (18)

and $e$ is the electric field, and the particle number,

$$N := \text{Tr} \gamma.$$

**Theorem.** The physically interesting stationary BdG solutions are critical points of the free energy

$$F_T(\eta, a) := E(\eta, a) - TS(\eta) - \mu N(\eta),$$  \hspace{1cm} (19)

where $S(\eta) = - \text{Tr}(\eta \ln \eta)$, the entropy, $N(\eta) := \text{Tr} \gamma$.

Since the BDG eqs are translation inv., the ground state energy and the number of particles are expected to be either 0 or $\infty$. 
Gauge (magnetic) translational invariance

The BdG eqs equations are invariant under the *gauge* transforms

\[ T_{\chi}^{\text{gauge}} : (\gamma, \alpha, a, \phi) \rightarrow (e^{i\chi} e^{-i\chi}, e^{i\chi} e^{i\chi}, a + \nabla \chi, \phi + \partial_t \phi) \]  

(20)

\[ \Rightarrow \text{ states related by a gauge transform are physically equiv.} \]

For \( a \neq 0 \), the simplest class of states are the gauge translationally invariant ones. (Translationally invariant states \( \Leftrightarrow a = 0 \).)

Gauge (magnetically) transl. invariant states are invariant under

\[ T_{bs} : (\eta, a) \rightarrow (T_{\chi s}^{\text{gauge}})^{-1} T_{s}^{\text{trans}} (\eta, a), \]  

(21)

for any \( s \in \mathbb{R}^2 \), where \( \chi_s(x) := \frac{b}{2} (s \wedge x) \) (modulo \( \nabla f \)).

The next result shows that, unlike the \( b = 0 \) translation invariant case, there are no non-normal magnetically translationally (MT)) invariant states.
Recall $\eta \sim (\gamma, \alpha)$. The BdG eqs have the following classes of stationary solutions which are the candidates for the ground state:

1. Normal states: $(\gamma, \alpha, a)$, with $\alpha = 0$ ($\iff \eta$ is diagonal).
2. Superconducting states: $(\gamma, \alpha, a)$, with $\alpha \neq 0$ and $a = 0$.
3. Mixed states: $(\gamma, \alpha, a)$, with $\alpha \neq 0$ and $a \neq 0$.

For $a = 0$, the existence of superconducting and normal, translationally invariant solutions is proven by Hainzl, Hamza, Seiringer, Solovej.

**Theorem.** MT-invariance $\implies$ normality ($\alpha = 0$).

**Corollary.** Mixed states break the magnetic translational symmetry.

From now on, $d = 2$, i.e. we consider the cylinder geometry.
Results at a glance

Theorem [Li Chen-IMS] Let $b > 0$. Then $\exists 0 \leq T'_c(b) \leq T''_c(b)$ s.t.

- the energy minimizing states with $T > T''_c(b)$ are normal;
- the energy minimizing states with $T < T'_c(b)$ are mixed.
Normal states and symmetry breaking

**Theorem.** Drop the exchange term $\nu \# \gamma$ and let $|\int \nu|$ be small. Then $\forall T, b > 0$

(i) the BdG equations have a unique mt-invariant solution.

(ii) mt-invariance $\iff$ normality ($\alpha = 0$) $\iff$ $(\gamma_T, b, 0, a_b)$, where

$$\gamma_{Ta} \text{ solves } \gamma = f\left(\frac{1}{T} h_{\gamma,a}\right),$$

with $f(h) = (1 + e^h)^{-1}$ the Fermi-Dirac distribution, and $a_b(x) = \text{magnetic potential with a constant magn. field } b$.

**Theorem.** Suppose that $\nu \leq 0$, $\nu \neq 0$. Then,

- for $T > 0$ and $b$ large, the normal solution is stable,

- for $b$ and $T$ small, the normal solution is unstable.

**Open problem.** Are minimizers among normal states MT invariant?
Mixed states

Let $\mathcal{L} = r(\mathbb{Z} + \tau\mathbb{Z})$, where $\tau \in \mathbb{C}$, $\text{Im} \tau > 0$. We define

- **Vortex lattice**: $T_{s}^{\text{trans}}(\eta, a) = T_{\chi_{s}}^{\text{gauge}}(\eta, a)$, for every $s \in \mathcal{L}$ and a co-cycle $\chi_{s} : \mathcal{L} \times \mathbb{R}^{2} \to \mathbb{R}$, and $\alpha \neq 0$.

(The condition $\alpha \neq 0$ rules out that $(\eta, a)$ is magnetically translationally invariant and therefore a normal state.)

The magnetic flux is quantized ($\Omega_{\mathcal{L}}$ is a fundamental cell of $\mathcal{L}$):

$$\frac{1}{2\pi} \int_{\Omega_{\mathcal{L}}} \text{curl} \ a = c_{1}(\chi) \in \mathbb{Z}.$$

A vortex lattice solution is formed by **magnetic vortices**, arranged in a (mesoscopic) lattice $\mathcal{L}$.

Magnetic vortices are **localized finite energy** solutions of a fixed degree, they are excitations of the homogeneous ground state.
Magnetic vortices and vortex lattices
Existence of vortex lattices

Theorem

(i) \( \forall n \text{ and } \mathcal{L}, \exists \text{ a solution } (\eta, a) \text{ of the BdG eqs satisfying} \)

\[ T_{s}^{\text{trans}}(\eta, a) = \hat{T}_{\chi_s}^{\text{gauge}}(\eta, a), \forall s \in \mathcal{L}, \]

\[ \int_{\mathcal{L}} \text{curl } a = 2\pi n; \]

(ii) This solution minimizes the free energy \( F_{T} \) on \( \Omega_{\mathcal{L}} \) for \( c_{1} = n; \)

(iii) For \( \nu \leq 0, \nu \neq 0 \) and \( T \) and \( b \) sufficiently small, this solution is a vortex lattice (i.e. \( \alpha \neq 0 \));

(iv) For \( n > 1 \), there is a finer lattice, for which this solution is equivariant and \( c_{1} = 1 \).
Vortex Lattice. Experiment
Summary

- considered the Bogolubov-de Gennes equations, which are equivalent to the BCS theory of superconductivity
- introduced the key stationary solutions of BdG eqs, the competitors for the ground state: normal, superconducting and mixed (or intermediate) states
- described a rough phase diagram in the temperature - magnetic field plane
- discussed the magnetic translation symmetry and its spontaneous breaking
- presented an important class of the mixed states - the vortex lattices - demonstrating the symmetry breaking
Thank-you for your attention
Ginzburg-Landau Equations

Discovery of the vortex lattices are a crown achievement of theory of superconductivity. They were predicted by A. A. Abrikosov on the basis of the Ginzburg-Landau equations:

\[-\Delta_a \psi = \kappa^2 (1 - |\psi|^2) \psi\]
\[\text{curl}^* \text{curl} a = \text{Im}(\bar{\psi} \nabla_a \psi)\]

where \((\psi, a) : \mathbb{R}^d \rightarrow \mathbb{C} \times \mathbb{R}^d, d = 2, 3, \nabla_a = \nabla - ia, \Delta_a = \nabla_a^2,\) the covariant derivative and covariant Laplacian, respectively, and \(\kappa\) is the Ginzburg-Landau (material) constant.

These equations describe equilibrium states of superconductors (mesoscopically) and of the \(U(1)\) Yang-Mills-Higgs model of particle physics.

Formally, they approximate the stationary BdG in the mesoscopic regime.
**Superconductivity**: \( \psi : \mathbb{R}^d \rightarrow \mathbb{C} \) is called the *order parameter*; \( |\psi|^2 \) gives the density of (Cooper pairs of) superconducting electrons. \( a : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is the magnetic potential. \( \text{Im}(\bar{\psi} \nabla_a \psi) \) is the superconducting current.

**Particle physics**: \( \psi \) and \( a \) are the Higgs and \( U(1) \) gauge (electro-magnetic) fields, respectively. (Part of Weinberg - Salam model of electro-weak interactions/a standard model.)

**Time-dependent equations**: The corresponding time-dependent equations are complex nonlinear Schrödinger and nonlinear (relativistic) wave equations coupled to a Maxwell equation.

**Key problem**: Dynamical stability of Abrikosov lattices.
GLE on Riemann surfaces

Abrikosov vortex lattices $\iff \mathcal{L}$–equivariant functions and vector fields (one forms) $\iff$ sections and connections of the line bundle over a complex torus, $\mathbb{T} = \mathbb{C}/\mathcal{L}$.

$\implies$ reformulate the Ginzburg-Landau equations as equations on $\mathbb{T}$:

\begin{align}
\Delta_a \psi &= \kappa^2 (|\psi|^2 - 1) \psi, \\
\dd^* da &= \text{Im}(\bar{\psi} \nabla_a \psi).
\end{align}

Here $\psi$ is a section and $a$, a connection one-form on a $U(1)$ line bundle $L \to \mathbb{T}$, $\Delta_a = \nabla^*_a \nabla_a$, $\nabla_a$ and $\nabla^*_a$ are the covariant derivative and its adjoint, and $\dd$ and $\dd^*$ are the exterior derivative and its adjoint, which replace curl and curl$^*$.

The complex torus, $\mathbb{T}$ is one of the simplest Riemann surfaces, but we can consider (23) on an arbitrary Riemann surface $X$. 
Returning to the covering space

By the key uniformization theorem for Riemann surfaces, a Riemann surface $X$ of genus $\geq 2$ is of the form

$$X = \mathbb{H}/\Gamma,$$

for some discrete subgroup $\Gamma \subset PSL(2, \mathbb{R})$ (Fuchsian group) acting freely (i.e. without fixed points) on the Poincaré half-plane

$$\mathbb{H} := \{z \in \mathbb{C} : \text{Im} \, z > 0\}.$$

($\eta$ acts on $\mathbb{H}$ as $\gamma \, z = \frac{az+b}{cz+d}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \eta.$)

Lifting the GLEs to $\mathbb{H}$, it becomes analogous to the original GLEs but with $\mathbb{C}$ replaced by $\mathbb{H}$ and the lattice $\mathcal{L}$ by a Fuchsian group, $\Gamma$.

E.g. the $\mathcal{L}$—gauge invariance is replaced by $\Gamma$—gauge invariance.
Periodicity w.r.to $\Gamma$

Tiling of the hyperbolic plane with equilateral triangles

Rhombitriheptagonal tiling

Icosahedral honeycomb
Theorem

Given the hyperbolic metric $h^{\text{hyperb}} := |dz|^2/((\text{Im } z)^2$ on $\mathbb{H}$, and $n \in \mathbb{Z}$, the unique constant curvature connection on $\tilde{L}$ of the degree $n$ is given by

$$a^b = by^{-1}dx, \quad b = \frac{\pi n}{g - 1}.$$ 

It is equivariant with the automorphy factor

$$\rho(\gamma, z) = \left[\frac{cz + d}{cz + d}\right]^{-\frac{\pi n}{g - 1}}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R}). \quad (24)$$
Summary

- I gave a thumbnail sketch of key PDEs of quantum physics concentrating on the Bogolubov-de Gennes equations. The latter describe the remarkable quantum phenomenon of superconductivity.

- There are many fundamental questions about these equations which are completely open.

- I introduced the key special solutions of BdGeqs: normal, superconducting and mixed (or intermediate) states.

- An important class of the mixed states are the vortex lattices.

- I discussed recent results on existence and stability of the normal and vortex lattice states.
Thank-you for your attention
Stationary Bogoliubov-de Gennes equations

We consider stationary solutions to BdG eqs of the form

\[ \eta_t := T^{\text{gauge}}_{\chi(t)} \eta^*, \]  

(25)

with \( \eta^* \) and \( \dot{\chi} \equiv \mu \) independent of \( t \), \( \chi \) independent of \( x \), and \( a \) independent of \( t \) and \( \phi = 0 \). We have

Proposition

(25), with \( \eta^* \) and \( \dot{\chi} \equiv -\mu \) independent of \( t \), is a solution to (15) iff \( \eta^* \) solves the equation

\[ [H_{\eta a}, \eta] = 0, \]  

(26)

where

\[ H_{\eta a} := \begin{pmatrix} h_{\gamma a\mu} & v^\# \alpha \\ v^\# \alpha^* & -h_{\gamma a\mu} \end{pmatrix}, \quad h_{\gamma a\mu} := h_{\gamma a} - \mu. \]  

(27)
Stationary Bogoliubov-de Gennes equations

For any reasonable function $f$, solutions of the equation

$$\eta = f\left(\frac{1}{T} H_{\eta a}\right)$$

(28)

solve $[H_{\eta a}, \eta] = 0 \implies$ give stationary solutions of BdG eqs.

Physics:

$$f(h) = (1 + e^{h/T})^{-1} \quad \text{(the Fermi-Dirac distribution)} \quad (29)$$

Let $f^{-1} =: g'$. Then the stationary Bogoliubov-de Gennes equations can be written as

$$H_{\eta a} - T g'(\eta) = 0,$$

$$\text{curl}^* \text{ curl } a = j(\eta, a).$$

(30)  \hspace{2cm} (31)
Free energy

**Theorem**

*The stationary BdG eqs are the Euler-Lagrange equations for the free energy*

\[ F_T(\eta, a) := E(\eta, a) - TS(\eta) - \mu N(\eta), \]  

(32)

where \( S(\eta) = -\text{Tr}(\eta \ln \eta) \), the entropy, \( N(\eta) := \text{Tr} \gamma \), the particle number.
Lifting sections and connections to $\tilde{H}$

**Proposition.** A connection $\nabla_A = d - iA$ and a section $\Psi$ are in one-to-one correspondence with a one-form $\tilde{A}$ and function $\tilde{\Psi}$ on $\tilde{X} = \tilde{H}$, which are gauge $\Gamma-$invariant, i.e. satisfy the relations

$$\gamma^*\tilde{\Psi} = \rho_{\gamma} \tilde{\Psi}, \quad \gamma^*\tilde{A} = \tilde{A} + \rho_{\gamma}^{-1} d\rho_{\gamma}, \quad \forall \gamma \in \eta, \quad (33)$$

where $\gamma^*\tilde{\Psi}(z) = \Psi(\gamma z)$, etc., for some automorphy factor, $\rho_{\gamma}(z) \equiv \rho(\gamma, z)$, i.e. a map $\rho : \Gamma \times H \to U(1)$ satisfying the co-cycle relation

$$\rho(\gamma \cdot \delta, z) = \rho(\gamma, \delta z) \rho(\delta, z).$$
The existence of normal states

We give a key idea of the proof of existence of normal states with non-vanishing magnetic fields.

Recall: \((\eta, a)\) is a normal state \(\iff\) \(\alpha = 0\) (\(\eta\) is diagonal)

When \(\alpha = 0\), the BdG equations reduces to the equations for \(\gamma\) and \(a\):

\[
\gamma = g^\#(\frac{1}{T} h_{\gamma,a}), \quad \text{curl}^* \text{ curl } a = j(\gamma, a)
\] (34)

where, recall, \(j(\gamma, a)(x) := \frac{1}{2}[-i\nabla_a, \gamma](x, x)\).

We show that the second equation is automatically satisfied, i.e. the superconducting current vanishes, for \(a = a_b\) and \(\gamma\) is magnetically translation invariant.
The existence of normal states

We define $t^\text{mt}_s := t^\text{gauge}_s t^\text{trans}_s$, where $g_s(x) := \frac{b}{2} s \wedge x$,

$$t^\text{gauge}_\chi: \gamma \mapsto e^{i\chi} \gamma e^{-i\chi}, \quad t^\text{trans}_h: \gamma \mapsto U_h \gamma U_h^{-1},$$

for any sufficiently regular function $\chi : \mathbb{R}^d \to \mathbb{R}$, and any $h \in \mathbb{R}^d$. Let $t^\text{refl}$ be a conjugation by reflections.

Proposition

If a trace class operator $\tilde{\gamma}$ satisfies $t^\text{mt}_h \tilde{\gamma} = \tilde{\gamma}$, then $\tilde{\gamma}(x, x) = \tilde{\gamma}(0, 0)$ for all $x$. If, in addition, $t^\text{refl} \tilde{\gamma} = -\tilde{\gamma}$, then $\tilde{\gamma}(x, x) = 0$.

Proof.

Due to $t^\text{mt}_h \tilde{\gamma} = \tilde{\gamma}$, the integral kernel of $\tilde{\gamma}$ obeys $e^{i g_h(x)} \tilde{\gamma}(x + h, y + h) e^{-i g_h(y)} = \tilde{\gamma}(x, y)$. Taking $y = x$ gives $\tilde{\gamma}(x + h, x + h) = \tilde{\gamma}(x, x)$, which implies $\tilde{\gamma}(x, x) = \tilde{\gamma}(0, 0)$. $t^\text{refl} \tilde{\gamma} = -\tilde{\gamma}$ implies $\tilde{\gamma}(-x, -y) = -\tilde{\gamma}(x, y)$, which gives $\tilde{\gamma}(-x, -x) = -\tilde{\gamma}(x, x)$, which implies $\tilde{\gamma}(x, x) = \tilde{\gamma}(0, 0) = 0$. \qed
Recall that \( j(\gamma, a)(x) := \frac{1}{2}[-i\nabla_a, \gamma](x, x) \). Consider the operator \( \tilde{\gamma} := \frac{1}{2}[-i\nabla_{ab}, \gamma] \).

If \( \gamma \) is magnetically translation invariant, then so is \( \tilde{\gamma} \). If \( \gamma \) is even under the reflections, then \( \tilde{\gamma} \) is odd.

Applying the proposition above to \( \tilde{\gamma} = \frac{1}{2}[-i\nabla_{ab}, \gamma] \), where \( \gamma \) a magnetically translationally invariant and even trace class operator gives \( j(\gamma, a_b) = 0 \).

Since \( \text{curl}^* \text{curl} a_b = \text{curl}^* b = 0 \), this proves \( \text{curl}^* \text{curl} a_b = j(\gamma, a_b) \), which is the second equation in (34). \( \square \)