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Institute of Science and Technology

Stability of quantum many-body systems with point interactions

Robert Seiringer
IST Austria

Joint work with Thomas Moser
arXiv:1609.08342, Commun. Math. Phys. (in press)

Quantissima in the Serenissima II

Venice, August 21–25, 2017

Preface: Point Interactions

Point interactions are ubiquitously used in physics, as effective models whenever the range of the interparticle interactions is much shorter than other relevant length scales.

Examples: Nuclear physics, polaron models, cold atomic gases, ...

Roughly speaking, one tries to make sense of a formal Hamiltonian of the form

$$H = -\sum_{i=1}^{N} \frac{1}{2m_i}\Delta_{x_i} + \sum_{1\leq i<j\leq N} \gamma_{ij}\delta(x_i - x_j) \ , \qquad x_i \in \mathbb{R}^3$$

The problem is completely understood for $N = 2$, but there are many open questions for $N \geq 3$:

- Does there exist a suitable **self-adjoint Hamiltonian** modeling point interactions between pairs of particles?
- If yes, is it **stable**, i.e., bounded from below?

The $N = 2$ Problem

Separating the center-of-mass motion, one can rigorously define $-\Delta + \gamma\delta(x)$ via **self-adjoint extensions** of $-\Delta$ on $C_0^\infty(\mathbb{R}^3 \setminus \{0\})$.

There exists a one-parameter family of such extensions, denoted by h_α for $\alpha \in \mathbb{R}$, with

$$D(h_\alpha) = \left\{ \psi \in L^2(\mathbb{R}^3) \,|\, \hat{\psi}(p) = \hat{\phi}(p) + \frac{\xi}{p^2 + \mu},\ \phi \in H^2(\mathbb{R}^3),\ \int \hat{\phi} = \left(\alpha + 2\pi^2\sqrt{\mu}\right) \xi \right\}$$

for $\mu > 0$ and

$$(h_\alpha + \mu)\psi = (-\Delta + \mu)\phi$$

Functions in $D(h_\alpha)$ satisfy

$$\psi(x) \approx \left(\frac{2\pi^2}{|x|} + \alpha \right) \frac{\xi}{(2\pi)^{3/2}} + o(1) \qquad \text{as } |x| \to 0$$

hence $\alpha = -2\pi^2/a$ with a the **scattering length** of the pair interaction.

The $N = 2$ Problem, Continued

One checks that

$$\inf \operatorname{spec} h_\alpha = \begin{cases} 0 & \text{for } \alpha \geq 0 \\ -\left(\frac{\alpha}{2\pi^2}\right)^2 & \text{for } \alpha < 0 \end{cases}$$

Moreover, the **quadratic form** for the energy reads

$$\langle \psi | h_\alpha \psi \rangle = E_\alpha(\psi) = \langle \phi | \left(-\Delta + \mu\right) \phi \rangle - \mu \|\psi\|^2 + |\xi|^2 \left(\alpha + 2\pi^2 \sqrt{\mu}\right)$$

with

$$D(E_\alpha) = \left\{ \psi \in L^2(\mathbb{R}^3) \,|\, \hat{\psi}(p) = \hat{\phi}(p) + \frac{\xi}{p^2 + \mu},\ \phi \in H^1(\mathbb{R}^3),\ \xi \in \mathbb{C} \right\}$$

The Hamiltonians h_α can be obtained by a suitable **limiting procedure**, e.g., taking $R \to 0$ for $-\Delta + V_R(x)$ with

$$V_R(x) = -\left(\frac{\pi^2}{4} + 2\frac{R}{a}\right) \begin{cases} R^{-2} & \text{for } |x| \leq R \\ 0 & \text{for } |x| \geq R \end{cases}$$

Stability for $N > 2$

It is known that stability fails, in general, for $N \geq 3$, unless the particles are fermions. This is known as the **Thomas effect**. It is closely related to the **Efimov** effect.

For n-component fermions, only particles in different "spin" states interact. Instability problem persists for $n \geq 3$.

For **two-component fermions**, stability fails if the mass ratio m_1/m_2 for the two components is too large ($\gtrsim 13.6$) or too small ($\lesssim 1/13.6$).

For the $2+1$ problem, stability is known in the opposite mass ratio regime. The general $N+M$ problem is open, however!

We consider here the simplest many-body problem, namely the $N+1$ **problem**, formally defined by

$$H = -\frac{1}{2m}\Delta_{x_0} - \frac{1}{2}\sum_{i=1}^{N}\Delta_{x_i} + \gamma\sum_{i=1}^{N}\delta(x_0 - x_i)$$

acting on wave functions $\psi(x_0, x_1, \ldots, x_N)$ **antisymmetric** in $(x_1, \ldots, x_N)$.

The Model, Part 1

Our model is defined via a **quadratic form** $\mathcal{F}_\alpha$ with domain

$$D(\mathcal{F}_\alpha) = \left\{ \psi = \phi + \mathcal{G}\xi \,|\, \phi \in H^1(\mathbb{R}^3) \otimes H^1_{\rm as}(\mathbb{R}^{3N}),\ \xi \in H^{1/2}(\mathbb{R}^3) \otimes H^{1/2}_{\rm as}(\mathbb{R}^{3(N-1)}) \right\}$$

where $\mathcal{G}(k_0, k_1, \dots, k_N) = \left(\frac{1}{2m} k_0^2 + \frac{1}{2} \sum_{i=1}^N k_i^2 + \mu \right)^{-1}$ and $\mathcal{G}\xi$ is short for the function with Fourier transform

$$\widehat{\mathcal{G}\xi}(k_0, k_1, \dots, k_N) = \mathcal{G}(k_0, k_1, \dots, k_N) \sum_{i=1}^N (-1)^{i+1} \hat{\xi}(k_0 + k_i, k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_N)$$

For $\psi \in D(\mathcal{F}_\alpha)$, we have

$$\mathcal{F}_\alpha(\psi) = \left\langle \phi \middle| -\frac{1}{2m}\Delta_{x_0} - \frac{1}{2}\sum_{i=1}^N \Delta_{x_i} + \mu \middle| \phi \right\rangle - \mu \|\psi\|^2$$
$$+ N \left(\frac{2m}{m+1} \alpha \|\xi\|^2_{L^2(\mathbb{R}^{3N})} + \mathcal{T}_{\rm diag}(\xi) + \mathcal{T}_{\rm off}(\xi) \right)$$

The Model, Part 2

where

$$\mathcal{T}_{\text{diag}}(\xi) = \int_{\mathbb{R}^{3(N-1)}} |\hat{\xi}(k_0, s, \vec{k})|^2 \mathcal{L}(k_0, s, \vec{k}) \, \mathrm{d}k_0 \, \mathrm{d}s \, \mathrm{d}\vec{k}$$

$$\mathcal{T}_{\text{off}}(\xi) = (N-1) \int_{\mathbb{R}^{3(N+1)}} \hat{\xi}^*(k_0 + s, t, \vec{k}) \hat{\xi}(k_0 + t, s, \vec{k}) \mathcal{G}(k_0, s, t, \vec{k}) \, \mathrm{d}k_0 \, \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}\vec{k}$$

with $\vec{k} = (k_1, \ldots, k_{N-2})$ and

$$\mathcal{L}(k_0, k_1, \ldots, k_{N-1}) = 2\pi^2 \left(\frac{2m}{m+1} \right)^{3/2} \left(\frac{k_0^2}{2(m+1)} + \frac{1}{2} \sum_{i=1}^{N-1} k_i^2 + \mu \right)^{1/2}$$

The **dangerous** term is $\mathcal{T}_{\text{off}}(\xi)$, which is unbounded from below and multiplied by $(N-1)$. It has to be controlled by $\mathcal{T}_{\text{diag}}(\xi)$.

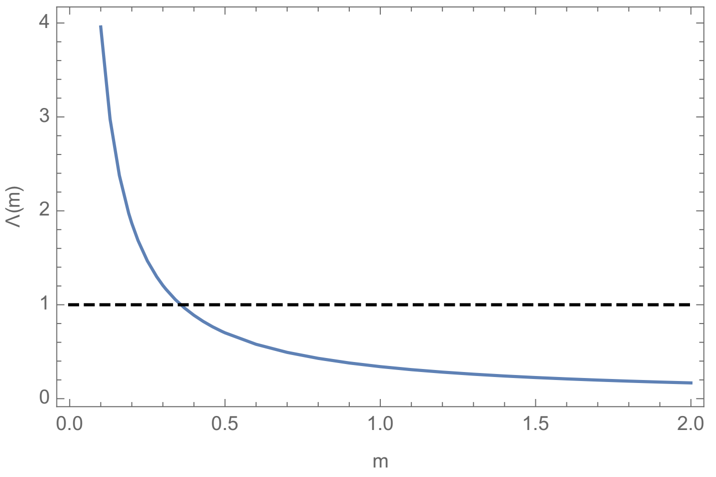
Note that even though all terms above depend on the choice of μ, $\mathcal{F}_\alpha(\psi)$ is actually independent of μ!

Main Result

THEOREM 1. *There exists* $\Lambda(m) > 0$, *independent of* N, *with* $\lim_{m\to\infty} \Lambda(m) = 0$, *such that*

$$\mathcal{T}_{\mathrm{off}}(\xi) \geq -\Lambda(m)\mathcal{T}_{\mathrm{diag}}(\xi)$$

A numerical evaluation of the explicit expression for $\Lambda(m)$ shows that $\Lambda(m) < 1$ for $m \geq 0.36$.



In particular, if m is such that $\Lambda(m) < 1$, then

$$\mathcal{F}_\alpha(\psi) \geq \begin{cases} 0 & \text{for } \alpha \geq 0 \\ -\left(\frac{\alpha}{2\pi^2(1-\Lambda(m))}\right)^2 \|\psi\|^2 & \text{for } \alpha < 0 \end{cases}$$

This lower bound is sharp as $m \to \infty$!

Recall that $\mathcal{F}_\alpha$ is known to be unbounded from below for any $N \geq 2$ for $m \leq 0.0735$. In particular, the **critical mass** for stability satisfies $0.0735 < m^* < 0.36$.

The Hamiltonian

For $\Lambda(m) < 1$, $\mathcal{F}_\alpha$ is closed and bounded from below, and thus gives rise to a **self-adjoint Hamiltonian** $\mathcal{H}_\alpha$. To define it, we need the positive operator Γ on $L^2(\mathbb{R}^3) \otimes L^2_{\rm as}(\mathbb{R}^{3(N-1)})$ defined by the quadratic form

$$\mathcal{T}_{\rm diag}(\xi) + \mathcal{T}_{\rm off}(\xi) = \langle \xi | \Gamma \xi \rangle$$

We have

$$D(\mathcal{H}_\alpha) = \Big\{ \psi = \phi + \mathcal{G}\xi \,|\, \phi \in H^2(\mathbb{R}^3) \otimes H^2_{\rm as}(\mathbb{R}^{3N}),\ \xi \in D(\Gamma),$$

$$\phi \restriction_{x_N = x_0} = \frac{(-1)^{N+1}}{(2\pi)^{3/2}} \left(\frac{2m\alpha}{m+1} + \Gamma \right) \xi \Big\}$$

and

$$(\mathcal{H}_\alpha + \mu)\,\psi = \left(-\frac{1}{2m}\Delta_{x_0} - \frac{1}{2}\sum_{i=1}^N \Delta_{x_i} + \mu \right) \phi$$

The Hamiltonian $\mathcal{H}_\alpha$ **commutes** with translations and rotations, and transforms under **scaling** as $U_\lambda \mathcal{H}_\alpha U_\lambda^* = \lambda^2 \mathcal{H}_{\lambda^{-1}\alpha}$.

Boundary Condition

For $\psi = \phi + \mathcal{G}\xi$, the **boundary condition**

$$\phi\restriction_{x_N=x_0} = \frac{(-1)^{N+1}}{(2\pi)^{3/2}}\left(\frac{2m\alpha}{m+1} + \Gamma\right)\xi$$

means that

$$\psi(x_0, x_1, \dots, x_N) \sim \left(\frac{1}{|x_0 - x_N|} - \frac{1}{a} + o(1)\right) \quad \text{as } |x_0 - x_N| \to 0.$$

More precisely: For any $\psi \in D(\mathcal{H}_\alpha)$,

$$\psi\left(R + \frac{r}{1+m}, x_1, \dots, x_{N-1}, R - \frac{mr}{1+m}\right)$$
$$= \left(\frac{2\pi^2}{|r|} + \alpha\right)\frac{2m}{m+1}\frac{(-1)^{N+1}}{(2\pi)^{3/2}}\xi(R, x_1, \dots, x_{N-1}) + \upsilon(R, x_1, \dots, x_{N-1}, r)$$

with $\lim_{r\to 0}\|\upsilon(\,\cdot\,, r)\|_{L^2(\mathbb{R}^{3N})} = 0$.

Tan Relations

For $\psi \in D(\mathcal{H}_\alpha)$, define the **contact**

$$\mathcal{C} = \left(\frac{2m}{m+1}\right)^2 N\|\xi\|^2$$

It shows up in a number of physically relevant quantities:

- The **two-particle density**

$$\int \varrho(R + \tfrac{r}{1+m}, R - \tfrac{mr}{1+m})\,\mathrm{d}R \approx \frac{\pi}{2}\left(\frac{1}{|r|^2} - \frac{2}{|r|a}\right)\mathcal{C} \quad \text{as } |r| \to 0$$

- The **momentum distributions**, $n_\uparrow(k) \approx n_\downarrow(k) \approx \mathcal{C}|k|^{-4}$ as $|k| \to \infty$
- $\frac{\partial}{\partial\alpha}\mathcal{F}_\alpha(\psi) = \frac{m+1}{2m}\mathcal{C}$ at fixed ψ ("adiabatic sweep theorem")
- The **energy**

$$\langle\psi|\mathcal{H}_\alpha\psi\rangle = \int_{\mathbb{R}^3}\left[\frac{k^2}{2m}\left(n_\uparrow(k) - \frac{\mathcal{C}}{|k|^4}\right) + \frac{k^2}{2}\left(n_\downarrow(k) - \frac{\mathcal{C}}{|k|^4}\right)\right]\mathrm{d}k - \frac{m+1}{2m}\mathcal{C}\alpha$$

Sketch of the Proof of the Main Theorem

- Separate center-of-mass motion to **eliminate** one degree of freedom; this leaves us with a problem of N fermions only.

- Identify the **negative part** of the operator corresponding to $\mathcal{T}_{\mathrm{off}}(\xi)$; this part is crucial, it is known that the inequality

$$\mathcal{T}_{\mathrm{off}}(\xi) \leq \mathcal{T}_{\mathrm{diag}}(\xi)$$

  fails for all $m > 0$ (and suitable ξ)

- Replace the factor $N-1$ by a sum over particles, using the **anti-symmetry**.

- Use a suitable version of the **Schur test** to bound the corresponding operator:

$$\|K\| \leq \sup_x \frac{1}{h(x)} \int |K(x,y)| h(y) \, \mathrm{d}y$$

  for any positive function h.

Conclusions

- We proved stability of the $N+1$ system of fermions with point interactions, for mass ratio $m \geq 0.36$ **independent** of N.
- We constructed the corresponding **self-adjoint** Hamiltonian.
- We showed the validity of the **Tan relations** for all functions in the domain of this Hamiltonian.
- Main **open problem:** Investigate the stability for the general $N+M$ system. For $N = M = 2$, numerical studies suggest stability in the whole parameter regime where the $2+1$ problem is stable.