Dynamics of Bose Einstein Condensates

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Based on joint work with Christian Brennecke

I. The Gross-Pitaevskii Limit

Hamiltonian: consider N bosons described by

$$H_N^{\text{trap}} = \sum_{j=1}^{N} \left[-\Delta_{x_j} + V_{\text{ext}}(x_j) \right] + \sum_{i<j}^{N} N^2 V(N(x_i - x_j))$$

with V_{ext} confining and $V \geq 0$, regular, radial, short range.

Scattering length: defined by zero-energy scattering equation

$$\left[-\Delta + \frac{1}{2} V(x) \right] f(x) = 0, \qquad f(x) \to 1$$

For $|x|$ large,

$$f(x) = 1 - \frac{a_0}{|x|} \quad \Rightarrow \quad a_0 = \text{scattering length of } V$$

By scaling

$$\left[-\Delta + \frac{N^2}{2} V(Nx) \right] f(Nx) = 0 \quad \Rightarrow \quad \frac{a_0}{N} = \text{scatt. length of } N^2 V(N.)$$

Ground state energy: [**Lieb-Seiringer-Yngvason, '00**] proved

$$\lim_{N\to\infty} \frac{E_N}{N} = \min_{\varphi\in L^2(\mathbb{R}^3):\|\varphi\|=1} \mathcal{E}_{\text{GP}}(x)$$

with Gross-Pitaevskii energy functional

$$\mathcal{E}_{\text{GP}}(\varphi) = \int \left[|\nabla\varphi|^2 + V_{\text{ext}}|\varphi|^2 + 4\pi a_0|\varphi|^4\right] dx$$

Bose-Einstein condensation: [**Lieb-Seiringer, '02**] showed

$$\gamma_N^{(1)} \to |\varphi_0\rangle\langle\varphi_0|$$

where φ_0 minimizes $\mathcal{E}_{\text{GP}}$.

Warning: this does not mean that $\psi_N \simeq \varphi_0^{\otimes N}$. In fact

$$\frac{1}{N}\left\langle\varphi_0^{\otimes N}, H_N\,\varphi_0^{\otimes N}\right\rangle \simeq \int \left[|\nabla\varphi_0|^2 + V_{\text{ext}}|\varphi_0|^2 + \frac{\widehat{V}(0)}{2}|\varphi_0|^4\right] dx$$

Correlations are crucial!

II. Time-evolution of BEC

Theorem [Brennecke - S., '17]: Let $\psi_N \in L^2_s(\mathbb{R}^{3N})$ such that

$$\begin{cases} a_N & := 1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \to 0 \\ b_N & := \left| \frac{1}{N} \langle \psi_N, H_N^{\text{trap}} \psi_N \rangle - \mathcal{E}_{\text{GP}}(\varphi_0) \right| \to 0 \end{cases} \qquad \text{as } N \to \infty$$

Let

$$H_N = \sum_{j=1}^N -\Delta_{x_j} + \sum_{i<j}^N N^2 V(N(x_i - x_j))$$

and $\psi_{N,t} = e^{-iH_N t}\psi_N$. Then, for all $t \in \mathbb{R}$,

$$1 - \langle \varphi_t, \gamma_{N,t}^{(1)} \varphi_t \rangle \leq C(a_N + b_N + N^{-1}) \exp(c \exp(c|t|))$$

where φ_t solves **time-dependent Gross-Pitaevskii** equation

$$i\partial_t \varphi_t = -\Delta \varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t$$

with initial data $\varphi_{t=0} = \varphi_0$.

Remark: result immediately implies

$$\text{Tr}\,\left|\gamma^{(1)}_{N,t} - |\varphi_t\rangle\langle\varphi_t|\right| \leq C(a_N + b_N + N^{-1})^{1/2} \exp(c \exp(c|t|))$$

Remark: if ψ_N is ground state of trapped systems we expect (in some cases, we know; see next talk) that $a_N, b_N \simeq N^{-1}$.

Alternative statement: let $\psi_N \in L^2_s(\mathbb{R}^{3N})$, $\varphi \in L^2(\mathbb{R}^3)$ s.t.

$$\begin{cases} a_N := \text{Tr}\,\left|\gamma^{(1)}_N - |\varphi\rangle\langle\varphi|\right| \to 0 \\ b_N := \left|\frac{1}{N}\langle\psi_N, H_N \psi_N\rangle - \int \left[|\nabla\varphi|^2 + 4\pi a_0 |\varphi|^4\right] dx\right| \to 0 \end{cases}$$

Let $\psi_{N,t} = e^{-iH_N t}\psi_N$. Then

$$1 - \langle\varphi_t \gamma^{(1)}_{N,t} \varphi_t\rangle \leq C(a_N + b_N + N^{-1}) \exp(c \exp(c|t|))$$

where φ_t solves GP equation with data $\varphi_{t=0} = \varphi$.

Previous works:

[**Erdős-S.-Yau, '06-'08**]: BBGKY approach, no rate. Simplification of parts of proof due to [**Klainerman-Machedon '07**], [**Chen-Hainzl-Pavlovic-Seiringer, '13**].

[**Pickl, '10**]: alternative approach, uncontrolled rate.

[**Benedikter-de Oliveira-S. '12**]: precise bounds on rate, approximately coherent initial data in Fock space.

Related results on mean-field dynamics, among others by **Adami, Ammari, Bardos, Breteaux, T. Chen, X. Chen, Erdős, Falconi, Fröhlich, Ginibre, Golse, Grillakis, Hepp, Holmer, Kirkpatrik, Knowles, Kuz, Lewin, Liard, Machedon, Margetis, Mauser, Mitrouskas, Nam, Napiorkowski, Nier, Pavlovic, Pawilowski, Petrat, Pickl, Pizzo, Rodnianski, Rougerie, S., Spohn, Staffilani, Teta, Velo, Yau**

III. Ideas from the proof

Orthogonal excitations: for $\psi_N \in L^2_s(\mathbb{R}^{3N})$ and $\varphi \in L^2(\mathbb{R}^3)$, write

$$\psi_N = \alpha_0 \varphi^{\otimes N} + \alpha_1 \otimes_s \varphi^{\otimes(N-1)} + \alpha_2 \otimes_s \varphi^{\otimes(N-2)} + \cdots + \alpha_N$$

where $\alpha_j \in L^2_{\perp\varphi}(\mathbb{R}^3)^{\otimes_s j}$.

As in [**Lewin-Nam-Serfaty-Solovej, '12**], [**Lewin-Nam-S. '15**], we define the unitary map

$$U_\varphi : L^2_s(\mathbb{R}^{3N}) \to \mathcal{F}^{\leq N}_{\perp\varphi} = \bigoplus_{j=0}^{N} L^2_{\perp\varphi}(\mathbb{R}^3)^{\otimes_s j}$$
$$\psi_N \to U\psi_N = \{\alpha_0, \alpha_1, \ldots, \alpha_N\}$$

Remark: $\psi_N = U^*_\varphi \xi_N$ exhibits BEC in $\varphi \in L^2(\mathbb{R}^3)$ if and only if $\xi_N \in \mathcal{F}^{\leq N}_{\perp\varphi}$ has small number of particles.

Evolution of BEC: define **excitation vector** $\widetilde{\xi}_{N,t} \in \mathcal{F}_{\perp\varphi_t}^{\leq N}$ through

$$e^{-iH_N t} U_{\varphi_0}^* \xi_N = U_{\varphi_t}^* \widetilde{\xi}_{N,t}$$

In other words,

$$\widetilde{\xi}_{N,t} = \widetilde{\mathcal{W}}_{N,t} \xi_N$$

with **fluctuation dynamics**

$$\widetilde{\mathcal{W}}_{N,t} = U_{\varphi_t} e^{-iH_N t} U_{\varphi_0}^* : \mathcal{F}_{\perp\varphi_0}^{\leq N} \to \mathcal{F}_{\perp\varphi_t}^{\leq N}$$

Need to show

$$\langle \widetilde{\xi}_{N,t}, \mathcal{N} \widetilde{\xi}_{N,t} \rangle = \langle \xi_N, \widetilde{\mathcal{W}}_{N,t}^* \mathcal{N} \widetilde{\mathcal{W}}_{N,t} \xi_N \rangle \leq C_t$$

Problem: we are neglecting **correlations**!

Need to modify fluctuation dynamics!

Idea from [Benedikter-de Oliveira-S. '12]: interested in evolution of approximately **coherent initial data**:

$$e^{-i\mathcal{H}_N t} W_0\, \xi_N = W_t\, \widetilde{\xi}_{N,t}, \qquad \text{with } W_t = \text{Weyl operator}$$

Describe correlations through **Bogoliubov transformations**

$$\widetilde{T}_t = \exp\left[\frac{1}{2}\int dxdy\, \left(\eta_t(x;y)a_x^* a_y^* - \text{h.c.}\right)\right]$$

Define **modified excitation vector** $\xi_{N,t}$ through

$$e^{-i\mathcal{H}_N t} W_0\, \widetilde{T}_0\, \xi_N = W_t\, \widetilde{T}_t\, \xi_{N,t}$$

With choice $w = 1 - f$ and

$$\widetilde{\eta}_t(x;y) = -Nw(N(x-y))\varphi_t(x)\varphi_t(y) \qquad \left(\simeq -\frac{a_0}{|x-y|}\varphi_t(x)\varphi_t(y)\right)$$

in was possible to show that

$$\langle \xi_{N,t}, \mathcal{N} \xi_{N,t}\rangle \le C_t.$$

Goal: apply similar idea for N-**particles** data.

Problem: Bogoliubov transf. do not leave $\mathcal{F}_{\perp\varphi_t}^{\leq N}$ invariant.

Modified fields: on $\mathcal{F}_{\perp\varphi_t}^{\leq N}$, we define, for $f \in L^2_{\perp\varphi_t}(\mathbb{R}^3)$,

$$b^*(f) = a^*(f)\sqrt{\frac{N-\mathcal{N}}{N}}, \qquad b(f) = \sqrt{\frac{N-\mathcal{N}}{N}}\,a(f)$$

Remark

$$U^*_{\varphi_t} b^*(f) U_{\varphi_t} = a^*(f)\frac{a(\varphi_t)}{\sqrt{N}}$$

Generalized Bogoliubov transformations: define

$$T_t = \exp\left[\frac{1}{2}\int dxdy\,\left(\eta_t(x;y)b^*_x b^*_y - \text{h.c.}\right)\right]$$

Then $T_t : \mathcal{F}_{\perp\varphi_t}^{\leq N} \to \mathcal{F}_{\perp\varphi_t}^{\leq N}$, if η_t **orthogonal** to φ_t in both variables.

Modified fluctuation dynamics: let

$$\mathcal{W}_{N,t} = T_t^* \, U_{\varphi_t} \, e^{-iH_N t} \, U_{\varphi_0}^* \, T_0 : \mathcal{F}_{\perp \varphi_0}^{\leq N} \to \mathcal{F}_{\perp \varphi_t}^{\leq N}$$

Generator: define $\mathcal{G}_{N,t}$ such that

$$i\partial_t \mathcal{W}_{N,t} = \mathcal{G}_{N,t} \mathcal{W}_{N,t}$$

We have

$$\mathcal{G}_{N,t} = (i\partial T_t^*) T_t + T_t^* \left[(i\partial_t U_{\varphi_t}) U_{\varphi_t}^* + U_{\varphi_t} H_N U_{\varphi_t}^* \right] T_t$$

The contribution $(i\partial_t T_t^*) T_t$ is harmless.

We focus on the second term. Using rules

$$\begin{aligned} U_{\varphi_t} \, a^*(f) a(g) \, U_{\varphi_t}^* &= a^*(f) a(g) \\ U_{\varphi_t} \, a^*(\varphi_t) a(\varphi_t) U_{\varphi_t}^* &= N - \mathcal{N} \\ U_{\varphi_t} \, a^*(f) a(\varphi_t) \, U_{\varphi_t}^* &= a^*(f) \sqrt{N - \mathcal{N}} = \sqrt{N} \, b^*(f) \\ U_{\varphi_t} \, a^*(\varphi_t) a(f) \, U_{\varphi_t}^* &= \sqrt{N - \mathcal{N}} a(f) = \sqrt{N} \, b(f) \end{aligned}$$

we find

$$(i\partial_t U_{\varphi_t})U^*_{\varphi_t} + U_{\varphi_t} H_N U^*_{\varphi_t} = \sum_{j=1}^4 \mathcal{L}^{(j)}_{N,t}$$

with (roughly)

$$\mathcal{L}^{(1)}_{N,t} = \sqrt{N}\, b((N^3 V(N.) w(N.) * |\varphi_t|^2)\varphi_t) + \text{h.c.}$$

$$\begin{aligned}\mathcal{L}^{(2)}_{N,t} = & \int \nabla_x a^*_x \nabla_x a_x + \int dx \left[N^3 V(N.) * |\varphi_t|^2\right](x)\, b^*_x b_x \\ & + \int dxdy N^3 V(N(x-y))\varphi_t(x)\bar{\varphi}_t(y) b^*_x b_y \\ & + \frac{1}{2}\int dxdy N^3 V(N(x-y))\left[\varphi_t(x)\varphi_t(y) b^*_x b^*_y + \text{h.c.}\right]\end{aligned}$$

$$\mathcal{L}^{(3)}_{N,t} = \int dxdy N^{5/2} V(N(x-y))\left[\varphi_t(y) b^*_x a^*_y a_x + \text{h.c.}\right]$$

$$\mathcal{L}^{(4)}_{N,t} = \frac{1}{2}\int dxdy N^2 V(N(x-y)) a^*_x a^*_y a_y a_x$$

We find

$$\mathcal{G}_{N,t} = C_{N,t} + \mathcal{H}_N + \mathcal{E}_{N,t}$$

with

$$\mathcal{H}_N = \int \nabla_x a_x^* \nabla_x a_x + \frac{1}{2} \int dx dy \, N^2 V(N(x-y)) a_x^* a_y^* a_y a_x$$

and, for any $\delta > 0$, a $C > 0$ s.t.

$$\begin{aligned} \pm \mathcal{E}_{N,t} &\leq \delta \mathcal{H}_N + C(\mathcal{N} + 1) \\ \pm \left[i\mathcal{N}, \mathcal{E}_{N,t} \right] &\leq \delta \mathcal{H}_N + C(\mathcal{N} + 1) \\ \pm \dot{\mathcal{E}}_{N,t} &\leq \delta \mathcal{H}_N + C(\mathcal{N} + 1) \end{aligned}$$

Control of $\mathcal{N}$: by **Gronwall**, we conclude

$$\langle \xi_N, \mathcal{W}_{N,t}^* \mathcal{N} \, \mathcal{W}_{N,t} \, \xi_N \rangle \leq C_t \langle \xi_N, (\mathcal{N} + \mathcal{H}_N) \, \xi_N \rangle$$

With assumptions on initial data, theorem follows.

Main challenge: action of Bogoliubov transf. $\widetilde{T}_t$ is explicit, i.e.

$$\widetilde{T}_t \, a^*(f) \, \widetilde{T}_t = a^*(\cosh_{\eta_t}(f)) + a(\sinh_{\eta_t}(\bar{f}))$$

For generalized Bogoliubov transformations, **no explicit formula** is available.

Instead, we have to **expand**

$$T_t^* \, a^*(f) \, T_t = \sum_{n \geq 0} \frac{1}{n!} \mathrm{ad}^{(n)}(a^*(f))$$