Heat full statistics: heavy tails and fluctuations control

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joint work with T.Benoist, R. Raquépas

1. Full statistics and quantum fluctuation relations
2. Heat fluctuations: classical VS quantum
3. Mathematical settings and results
   - Bounded perturbations
   - Unbounded perturbations

Full statistics -confined systems

Full (counting) Statistics [Lesovik,Levitov '93][Levitov, Lee,Lesovik '96]

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Confined systems: $(\mathcal{H}, H, \rho)$ $dim\mathcal{H} < \infty$
Given an observable A: $A = \sum_j a_j P_{a_j}$ where $a_j \in \sigma(A)$ P_{e_j} associated spectral projections
At time 0 we measure A with outcome a_j with probability $\mathrm{tr}(\rho P_{a_j})$. Then the reduced state is

$$\frac{1}{\mathrm{tr}(\rho P_{a_j})} P_{a_j} \rho P_{a_j}.$$

Let evolve for time t, and measure again. The outcome will be a_k with probability

$$\frac{1}{\mathrm{tr}(\rho P_{a_j})} \mathrm{tr}(e^{-\mathrm{i}tH} P_{a_j} \rho P_{a_j} e^{\mathrm{i}tH} P_{a_k})$$

Full statistics -confined systems

Joint probability of measuring a_j at time 0 and a_k at time t is:

$$\mathrm{tr}(e^{-\mathrm{i}tH}P_{a_j}\rho P_{a_j}e^{\mathrm{i}tH}P_{e_k})$$

Full (counting) statistic is the atomic probability measure on $\mathbb{R}$ defined by

$$\mathbb{P}_t(\phi) = \sum_{a_k - a_j = \phi} \mathrm{tr}(e^{-\mathrm{i}tH}P_{a_j}\rho P_{a_j}e^{\mathrm{i}tH}P_{a_k})$$

(probability distribution of the change of A measured with the protocol above)

Full statistics and fluctuation relations

Quantum extention of classical fluctuation relations
Classical case:[Evans-Cohen-Morris'93] [Evans-Searls '94] [Gallavotti-Cohen'94]

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Classical case:[Evans-Cohen-Morris'93] [Evans-Searls '94] [Gallavotti-Cohen'94]
Quantum case:
Definition: $(\mathcal{H}, H, \omega)$ is TRI iff there exists an anti-linear $*$-automorphism, $\Theta^2 = \mathbb{1}$, $\tau_t \circ \Theta = \Theta \circ \tau_{-t}$ and $\omega(\Theta(A)) = \omega(A^*)$.
$A = S$ entropy

Proposition (Kurchan '00, Tasaki-Matsui '03)

Assume $(\mathcal{H}, H, \rho)$ is TRI, (and $\rho = \mathrm{e}^{-\beta \cdot H}/\mathrm{tr}(\mathrm{e}^{-\beta H})$.) .
Set $\bar{\mathbb{P}}_t(\underline{\phi}) := \mathbb{P}_t(-\phi)$.

Then for any ϕ in $\mathbb{R}$,
$$\frac{\mathrm{d}\bar{\mathbb{P}}_t}{\mathrm{d}\mathbb{P}_t}(\phi) = \mathrm{e}^{-t\phi}.$$

Heat fluctuations: classical VS quantum

[1]Given a classical observable C and an intial state ρ, we call C-statistics the probability measure $\mathbb{P}_C$ such that $\int f(s)\mathrm{d}\mathbb{P}_C(s) = \int f(C)\mathrm{d}\rho$ for all $f \in \mathcal{B}(\mathbb{R})$

Heat fluctuations: classical VS quantum

Classical system

$(\mathcal{M}, H, \rho) \qquad H = H_0 + V$

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Full statistics are equivalent to the law $\mathbb{P}_{\triangle A_t}$ associated to $\triangle A_t := A_t - A$. [1]

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Energy conservation: $\triangle H_{0,t} = H_{0,t} - H_0 = V_t - V$ as function on $\mathcal{M}$ which yields

$$\mathbb{P}_{\triangle H_{0,t}} = \mathbb{P}_{\triangle V_t}$$

In particular if V is bounded by C: $\sup_t |\triangle H_{0,t}| < 2C$ and

$$supp(\mathbb{P}_{\triangle H_{0,t}}) \text{ bounded}$$

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Quantum system

$(\mathcal{H}, H, \rho) \qquad H = H_0 + V$

Energy conservation:

$H_{0,t} - H_0 = V_t - V$ as operators on $\mathcal{H}$
implies equality of spectral measures but in general

$$\mathbb{P}_{H_0,t} \neq \mathbb{P}_{V,t}$$

Mathematical setting and results: bounded perturbations

General defintion

$(\mathcal{O}, \tau^t, \omega)$ C^*-dynamical system

$\tau^t = \tau_0^t + \mathrm{i}[-, V]$ and ω is a τ_0^t invariant state

$\pi_\omega : \mathcal{O} \to \mathcal{B}(\mathcal{H}_\omega)$ a GNS representation $\omega(A) = (\Omega_\omega, A\Omega_\omega)_{\mathcal{H}_\omega}$

Liouvillean: $\tau_0^t(A) = \mathrm{e}^{-\mathrm{i}tL}\pi_\omega(A)\mathrm{e}^{-\mathrm{i}tL}$ and $L\Omega_\omega = 0$

Definition

We define the *energy full statistics* (FS) measure for time t, denoted $\mathbb{P}_t$, to be the spectral measure for the operator

$$L + \pi_\omega(V) - \pi_\omega(\tau^t(V)),$$

with respect to the vector Ω_ω

Mathematical setting and results: bounded perturbation

Notions of regularity

(γA) $\quad t \to \tau_0^t(V)$ admits a bounded analytic extention to the strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \frac{1}{2}\gamma\}$.

(nD) $\quad t \to \tau_0^t(V)$ is n times norm-differentiable,

Theorem (Benoist, P., Raquépas 2017)

Let $(\mathcal{O}, \tau, \omega)$ be a C^*-dynamical system as above and $\mathbb{P}_t$ be the energy full statistics measure associated to a self-adjoint perturbation $V \in \mathcal{O}$. Then

$$(nD) \Rightarrow \sup_{t\in\mathbb{R}} \mathbb{E}_t[\Delta E^{2n+2}] < \infty.$$

$$(\gamma A) \Rightarrow \sup_{t\in\mathbb{R}} \mathbb{E}_t[\mathrm{e}^{\gamma|\Delta E|}] \leq 2\mathrm{e}^{2\gamma v_0}.$$

Corollary

Under the conditions of the previous theorem,

$$(nD) \Rightarrow \mathbb{P}_t(\tfrac{1}{t}|\Delta E| \geq R) \leq C_n(Rt)^{-2n+2}$$

$$(\gamma A) \Rightarrow \mathbb{P}_t(\tfrac{1}{t}|\Delta E| \geq R) \leq C_\gamma e^{-Rt}.$$

Regularity condition optimality: Fermi gas with impurity

$\mathcal{H} = \Gamma_a(\mathfrak{h})$ with $\mathfrak{h} = \mathbb{C} \oplus L^2(\mathbb{R}_+, \mathrm{d}e)$
$H_0 = \mathrm{d}\Gamma(h_0)$ with $h_0 = \varepsilon_0 \oplus \hat{e}$
$H = \mathrm{d}\Gamma(h_0 + |\psi_1\rangle\langle\psi_f| + |\psi_f\rangle\langle\psi_1|) = \mathrm{d}\Gamma(h_0) + a^*(\psi_1)a(\psi_f) + a^*(\psi_f)a(\psi_1)$
with $\psi_1 = 1 \oplus 0$, $\psi_f = 0 \oplus f$.

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Theorem (Benoist, P., Raquépas 2017)

For the above model the following are equivalent:

1. $\sup_{t\in\mathbb{R}} \mathbb{E}_t[\Delta E^{2n+2}] < \infty$;
2. *for a non-trivial time interval* $[t_1, t_2]$ $\int_{t_1}^{t_2} \mathbb{E}_t[\Delta E^{2n+2}]\mathrm{d}t < \infty$;
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For this model (nD) is equivalent to $\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{e}} f \in L^2(\mathbb{R}_+, \mathrm{d}e)$ is n times norm- differentiable i.e $f \in \mathrm{Dom}(\hat{e}^n)$

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Remark: decay of f controls how high energy frequencies contribute to the interaction (ultraviolet regularization)

Regularity condition optimality: bosonic systems

Bose gas with impurity

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Conditions: $f \in \mathrm{Dom}(\hat{e}) \cap \mathrm{Dom}(\hat{e}^{-1/2})$ and $\|h_0^{-1/2}\psi_f\| \neq \varepsilon_0^{1/2}$

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Theorem (Benoist, P., Raquépas 2017)

For the Bose gas with impurity model, assume $(\gamma A')$: $t \to \tau_0^t(V)\Omega$ admits a bounded analytic extention to the strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \frac{1}{2}\gamma\}$. Then

1. $\sup_{t\in\mathbb{R}} \mathbb{E}_t[e^{\gamma \Delta E}] < \infty$;

For this model $(\gamma A')$ is equivalent to $\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{e}} f \in L^2(\mathbb{R}_+, \mathrm{d}e)$ extends to an analitic function on the strip i.e $f \in Dom(e^{\frac{1}{2}\gamma\hat{e}})$

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1. $\sup_{t\in\mathbb{R}} \mathbb{E}_t[\Delta E^{2n+2}] < \infty$;
2. *for a non-trivial time interval* $[t_1, t_2]$ $\int_{t_1}^{t_2} \mathbb{E}_t[\Delta E^{2n+2}]\mathrm{d}t < \infty$;
3. (nD') $t \to \tau_0^t(V)\Omega$ *is n-times norm differentable (here* Ω *is the vacuum)*

For this model, (nD') is equivalent to $\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{e}} f \in L^2(\mathbb{R}_+, \mathrm{d}e)$ is n times differentiable in the norm sense i.e $f \in \mathrm{Dom}(\hat{e^n})$

van Hove Hamiltonians

$\mathcal{H} = \Gamma_s(\mathfrak{h})$ with $\mathfrak{h} = L^2(\mathbb{R}_+, \mathrm{d}e)$
$H_0 = \mathrm{d}\Gamma(e)$, $H = \mathrm{d}\Gamma(e) + a^*(f) + a(f)$
Conditions: $f \in \mathrm{Dom}(\hat{e}) \cap \mathrm{Dom}(\hat{e}^{-1/2})$.

Theorem (Benoist, P., Raquépas 2017)

For the van Hove bosonic models the following are equivalent:

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Theorem (Benoist, P., Raquépas 2017)

For the van Hove model the following are equivalent:

1. $\sup_{t\in\mathbb{R}} \mathbb{E}_t[e^{\gamma\Delta E}] < \infty$;
2. *for a non-trivial time interval* $[t_1(\gamma), t_2(\gamma)]$ $\int_{t_1(\gamma)}^{t_2(\gamma)} \mathbb{E}_t[e^{\gamma\Delta E}]\mathrm{d}t < \infty$;
3. $(\gamma A')$

For this model $(\gamma A')$ is equivalent to $\mathbb{R} \ni s \mapsto \mathrm{e}^{\mathrm{i}s\hat{\mathrm{e}}} f \in L^2(\mathbb{R}_+, \mathrm{d}e)$ extends to an analitic function on the strip i.e $f \in Dom(e^{\frac{1}{2}\gamma\hat{\mathrm{e}}})$

Summary

- Classical statistical mechanics models: heat fluctuations are controlled by the **interaction intensity**. Particularly, **no fluctuations exist when the interaction is bounded**.
- Quantum statistical mechanics models: in the two time measurement picture, heat fluctuations are controlled by a notion of regularity. Particularly, **large fluctuations may exists even if the interaction is bounded**.
  In concrete models, regularity notion translate in **contribtuon of high energy frequencies to the interaction** (UV regularization).

Thank you for your attention!