La Serenissima, 22 August, 2017

Stability of the superselection sectors of abelian quantum double models

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Joint work with

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[1]Based on work supported by the U.S. National Science Foundation under grant # DMS-1515850.

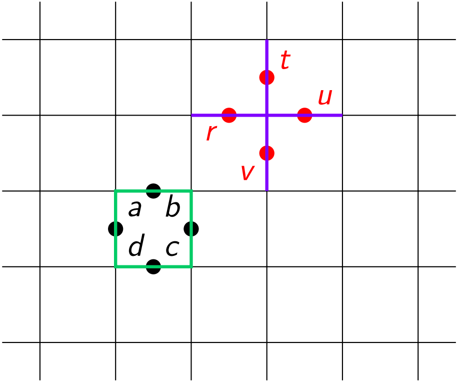
Outline

- Kitaev's quantum double models — toric-code model
- Ground states and excitations
- Stability of the vacuum state
- Stability of superselection sectors

Kitaev's quantum double models (Kitaev, 2003)

To save time we focus in these slides on the Toric Code model (TCM). There is a similar model for any finite group G ($G = \mathbb{Z}_2$ for TCM), and everything generalizes to arbitrary abelian G and many results also hold for non-abelian G.

$\mathcal{H}_x = \mathbb{C}^2$ for all $x \in \mathcal{E}(\mathbb{Z}^2)$, the edges of the square lattice, and we are interested in the infinite-volume model.



$$H = \sum_s (\mathbb{1} - A_s)$$
$$+ \sum_p (\mathbb{1} - B_p)$$

$$A_s = \sigma_r^1 \sigma_t^1 \sigma_u^1 \sigma_v^1$$

$$B_p = \sigma_a^3 \sigma_b^3 \sigma_c^3 \sigma_d^3$$

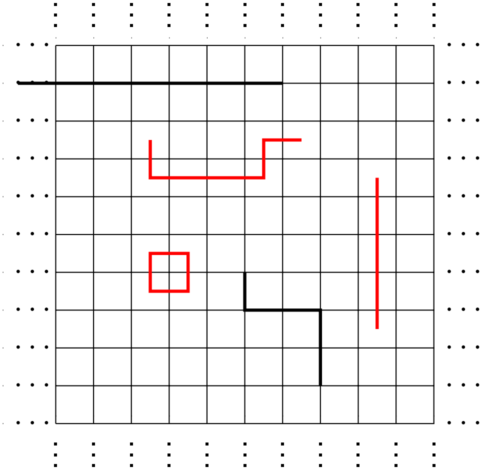
As shown by Kitaev, on a finite torus (periodic b.c.), or more generally on a surface of genus g, the TCM has 4^g frustration free ground states.

On $\mathbb{Z}^2$, the model has a unique frustration free ground state (Alicki-Fannes-Horodecki, 2007): there is a unique state ω_0 on the infinite lattice such that $\omega_0(\mathbb{1} - A_s) = \omega_0(\mathbb{1} - B_p) = 0$ for all stars s and plaquettes p of the infinite square lattice.

The spectrum of the GNS Hamiltonian of ω_0 is $\{0, 4, 8, 12, \ldots\}$, and the corresponding eigenstates are obtained from the vacuum vector Ω_0 by applying string (ribbon) operators of the form

$$F^e_\rho = \prod_{x\in\rho} \sigma^3_x, \quad F^\mu_\sigma = \prod_{x\in\sigma} \sigma^1_x,$$

where ρ and σ are finite open paths in the lattice and the dual lattice, respectively. The corresponding operators for closed paths leave Ω_0 invariant.



Lattice paths and dual paths. Each endpoint contributes energy 2 (one term in the Hamiltonian).

It is now clear that there are non-frustration-free ground states of the TCM by considering half-infinite paths and defining string/ribbon automorphisms. For example, consider the sequence of the lattice paths ρ_n, connecting $(0,0)$ with $(n,0)$ in a straight line. Then,

$$\tau_n^e(A) = F_{\rho_n}^e A F_{\rho_n}^e, \quad A \in \mathcal{A}_{\mathrm{loc}},$$

extends to an automorphism of the observable algebra and so does the limit $\tau^e(A) = \lim_{n\to\infty} \tau_n^e(A)$. (Naaijkens 2011). Then,

$$\omega_e = \omega_0 \circ \tau^e$$

is a pure state satisfying $\omega_e(\mathbb{1} - A_s) = \omega_e(\mathbb{1} - B_p) = 0$ for all stars s and plaquettes p, except for the star at the origin for which we have $\omega_e(\mathbb{1} - A_0) = 2$. The state ω_e does not depend on the choice of the half-infinite path ρ except for its starting point 0. This property is referred to as path independence.

In this way, one obtains a state with unit 'electric charge' for each starting point $x \in \mathbb{Z}^2$. By using dual paths, one can also define a family of 'magnetically' charged states ω_μ, and one can combine both by composing automorphisms and define $\omega_{e\mu}$. All these states are ground states in the sense that they satisfy

$$\lim_{\Lambda \uparrow \mathbb{Z}^2} \omega(A^*[H_\Lambda, A]) \geq 0, \text{ for all } A \in \mathcal{A}_{\mathrm{loc}}. \qquad (*)$$

Theorem (Cha-Naaijkens-N, 2016 (CMP, to appear))

The set of all solution of (*) for the TCM is the closed convex hull of the states $\omega_0, \omega_e, \omega_\mu, \omega_{e\mu}$, and all their translates and coherent superpositions of thereof.

Coherent superpositions only make sense for states within one of the 4 equivalence classes of states belonging to the same superselection sector.

Within each class the states are unitarily equivalent, but states from different classes are disjoint. The 4 classes are

$$K^0 = \{\omega_0\}, K^e, K^\mu, K^{e,\mu},$$

which correspond to: the vacuum state ω_0, which is translation invariant, and the linear span of all translates of ω_e, ω_μ, and $\omega_{e\mu}$, respectively.

What are superselection sectors?

Superselection sectors

In the framework of 'local quantum physics' (Doplicher-Haag-Roberts) a superselection sector is an equivalence class of representations of the observable algebra generated by composing certain types endopmorphisms with the vacuum representation These equivalence classes are typically labeled by the values of a type of conserved quantities of the theory, called charges.

Let $\pi_0 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_0)$ denote the GNS representation of the vacuum state ω_0. Then, for any endomorphism ρ of $\mathcal{A}$, $\pi_0 \circ \rho$ is another representation of $\mathcal{A}$, and we consider equivalence classed of such representations defined by unitary equivalence: $\pi_1 \sim \pi_0$ if there exists a unitary $U \in \mathcal{B}(\mathcal{H}_0)$ such that $\pi_1(A) = U^* \pi_0(A) U, A \in \mathcal{A}$.

To obtain something physically meaningful we have restrict to ρ that have (at least) two essential properties:

- Almost-locality in cones: we denote the set of cones in $\mathbb{Z}^2$ with opening angle α by $\mathcal{C}_\alpha$ and require of ρ that there is $\alpha \in (0, \pi)$ and $\Lambda \in \mathcal{C}_\alpha$, such that for all $\epsilon \in (0, \pi - \alpha)$, $k \geq 0$

$$\lim_{n\to\infty} n^k \sup_{A \in \mathcal{A}_{\Lambda(\epsilon)^c - n}, \|A\|=1} \|\rho(A) - A\| = 0$$

where '$-n$' denotes translation by n in the direction opposite to the forward direction of the axis of Λ, $\Lambda(\epsilon)$ is the cone obtained from Λ by widening its opening angle by ϵ.

- transportability with respect to the vacuum state: for any two cones $\Lambda, \Lambda' \in \mathcal{C}_\alpha$, and ρ (almost) localized in Λ, there is an equivalent ρ' (almost) localized in Λ'.

'Almost locality' is the quasi-local version of 'locality' employed by Doplicher-Haag-Roberts (1971-74) in algebraic QFT and the strict locality in cones used for the TCM by Naaijkens (2011). It is used in Cha's PhD thesis (2017) to treat perturbations of TCM.

Superselection sectors of the TCM

The superselection sectors of the TCM given as the equivalence classes of automorphisms localized in cones (Naaijkens 2011) is given by 4 classes of states equivalent to 4 classes of ground states $K^0, K^e, K^\mu, K^{e,\mu}$ and can be given the structure of the braided C^* tensor category of the representations of the quantum double $\mathcal{D}(G = \mathbb{Z}_2)$.

If we add a finite-energy condition, we can show that this structure is stable under uniformly small perturbations of the TCM.

In particular, the same type of anyons describe its low-energy excitations.

Quasi-locality and automorphic equivalence - the spectral flow

Consider perturbations are of the form

$$H_\Lambda(s) = H_\Lambda^{TCM} + s \sum_{X \subset \Lambda} \Phi(X).$$

with Φ an interaction such that for some $a > 0$

$$\|\Phi\|_a = \sup_{x,y \in \mathbb{Z}^2} e^{a|x-y|} \sum_{\substack{X \subset \mathbb{Z}^2 \\ x,y \in X}} \|\Phi(X)\| < \infty,$$

For what follows it will be important that H_Λ^{TCM} is frustration-free, gapped, and that its ground states satisfy a property called Local Topological Quantum Order (a consequence of the previously mentioned path-independence).

Stability of the superselection sectors

Theorem (Cha 2017, Cha-Naaijkens-N (in preparation))

There exists $s_0 > 0$ *such that for* $|s| \leq s_0$, *there exists an automorphism* α_s *with the following properties:*
(i) α_s *is the dynamics corresponding to a time-dependent short-range interaction* $\Psi(s)$
(ii) $\omega_0 \circ \alpha_s$ *is a translation invariant infinite volume ground states of the perturbed model, with a positive spectral gap;*
(iii) $K^k \circ \alpha_s$, *for* $k \in \{0, e, \mu, e\mu\}$, *describe the finite-energy superselection sectors of the perturbed model and are generated by almost localized automorphisms* $\tau_s^k = \alpha_s^{-1} \tau^k \circ \alpha_s$;
(iv) The set of superselection sectors of the perturbed model has the same braided (C^**-) fusion tensor category structure as TCM.*

The main tool in the proof, the automorphisms α_s, called the spectral flow, are constructed using Lieb-Robinson bounds Bachmann-Michalakis-N-Sims (2012).
Quasi-locality properties follow from the fact that α_s is the dynamics corresponding to a time-dependent short-range interaction $\Psi(s)$ with, for some $c > 0$,

$$\sup_{x,y\in\mathbb{Z}^2} e^{c|x-y|/(\log|x-y|)^2} \sum_{\substack{X\subset\mathbb{Z}^2\\ x,y\in X}} \sup_{|s|\leq s_0} \|\Phi(s,X)\| < \infty.$$

Lieb-Robinson bounds are used to prove this and, in turn, this structure implies that α_s satisfies a Lieb-Robinson bound, which is used to prove that $\tau_s = \alpha_s^{-1}\tau\circ\alpha_s$ is almost localized in a cone whenever τ is.
To define the fusion and braiding structure we use the framework of (bi-)asymptotias of Buchholz-Doplicher-Morchio-Roberts- Strocchi, 2007.

Comments and Outlook

- Exploiting quasi-locality is an essential ingredient in many recent results.
- Frustration-free models turn out to be a very useful class of examples.
- Stability of the superselection sectors also comes with stability of anyons (fusion and braiding).
- We need better techniques to prove spectral gaps in 2 and more dimensions.
- The nature and role 'edges states' for infinite systems with boundary needs mathematical investigation (non-abelian G).