Examples of particle creation at point sources via boundary conditions

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joint work with J. Schmidt, S. Teufel and R. Tumulka (Tübingen)

The minimal example

A simple model for a particle that can be emitted and absorbed by a source at $x_0 = 0 \in \mathbb{R}^3$ (Yafaev '92, Thomas '84).
On $L^2(\mathbb{R}^3) \oplus \mathbb{C}$ consider the operator

$$H = \begin{pmatrix} -\Delta_0^* & 0 \\ A & 0 \end{pmatrix}$$

where

- Δ_0^* is the adjoint of $\Delta_0 := (\Delta, H_0^2(\mathbb{R}^3 \setminus \{0\}))$
- $A : D(\Delta_0^*) \to \mathbb{C}$ extends the evaluation at $x = 0$:

$$A\psi = \lim_{r \to 0} \partial_r r\psi(r\omega)$$

on the domain $D(\Delta_0^*) \oplus \mathbb{C} \subset L^2(\mathbb{R}^3) \oplus \mathbb{C}$.

This operator is not symmetric.

The minimal example

It is well known that

$$D(\Delta_0^*) = H^2(\mathbb{R}^3) \oplus \operatorname{span}(f_\gamma)$$
$$f_\gamma(x) = -\frac{e^{-\gamma|x|}}{4\pi|x|}, \, \mathrm{Re}(\gamma) > 0.$$

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Let

$$B : D(\Delta_0^*) \to \mathbb{C}, \qquad \psi \mapsto -4\pi \lim_{|x|\to 0} |x|\psi(x),$$

integration by parts shows that

$$\begin{aligned}
&\langle \varphi, -\Delta_0^* \psi \rangle - \langle -\Delta_0^* \varphi, \psi \rangle \\
&= \int_0^\infty \int_{S^2} \Big(\Big(\partial_r^2 r \overline{\varphi}(r\omega) \Big) r\psi(r\omega) - r\overline{\varphi}(r\omega) \partial_r^2 r\psi(r\omega) \Big) \,\mathrm{d}r\mathrm{d}\omega \\
&= -\langle A\varphi, B\psi \rangle + \langle B\varphi, A\psi \rangle.
\end{aligned}$$

The minimal example

By

$$
\begin{aligned}
&\langle H\Phi, \Psi\rangle - \langle \Phi, H\Psi\rangle \\
&= -\langle A\varphi^{(1)}, B\psi^{(1)}\rangle + \langle B\varphi^{(1)}, A\psi^{(1)}\rangle + \langle A\varphi^{(1)}, \psi^{(0)}\rangle - \langle \varphi^{(0)}, A\psi^{(1)}\rangle
\end{aligned}
$$

H is symmetric on the domain

$$
D_{\mathrm{IBC}} = \Big\{ \Psi = (\psi^{(1)}, \psi^{(0)}) \in D(\Delta_0^*) \oplus \mathbb{C} : B\psi^{(1)} = \psi^{(0)} \Big\}.
$$

The condition $B\psi^{(1)} = \psi^{(0)}$ is a (co-dimension three) boundary condition at $x = 0$, we call this an *interior boundary condition* (IBC).

The minimal example

Proposition (Yafaev '92)

The operator H is self adjoint on the domain D_{IBC} and $H \geq 0$.

Proof.

Since H is symmetric on D_{IBC} it is enough to show that $(H + \lambda^2)\psi = g$ has a unique solution $\psi \in D_{\mathrm{IBC}}$ for $\lambda > 0$.
On the one-particle sector $\psi^{(1)} = \varphi + af_\lambda$, with $\varphi \in H^2(\mathbb{R}^3)$ and $B\psi^{(1)} = a = \psi^{(0)}$. Then

$$(-\Delta_0^* + \lambda^2)\psi^{(1)} = (-\Delta_0 + \lambda^2)\varphi$$

After solving $(H + \lambda^2)\psi = g$ for φ, we have the equation for $\psi^{(0)}$

$$\lambda^2\psi^{(0)} + \underbrace{Af_\lambda}_{=\lambda}\psi^{(0)} = g^{(0)} - A(-\Delta_0 + \lambda^2)^{-1}g^{(1)}$$

which is solvable for $\lambda > 0$. $\square$

A model on Fock space

An arbitrary number of particles can be created/annihilated by a source at the origin.

Let $\mathfrak{F}$ be the bosonic Fock space over $L^2(\mathbb{R}^3)$ and $\mathfrak{H}^n$ its n-particle sector. The singular set in the configuration space of n-particles is the set $\mathscr{C}^n$ with at least one particle at the origin.

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$$\Delta_0 = \Big(\Delta, H_0^2(\mathbb{R}^{3n} \setminus \mathscr{C}^n)\Big)$$

$$(B\psi)(x_1, \ldots, x_{n-1}) = -4\pi\sqrt{n} \lim_{|x_n|\to 0} |x_n|\psi(x_1, \ldots, x_n)$$

$$(A\psi)(x_1, \ldots, x_{n-1}) = \sqrt{n} \lim_{r\to 0} \partial_r r\psi(x_1, \ldots, x_{n-1}, r\omega)$$

and

$$D^{(n)} := \Big\{\psi \in D(\Delta_0^*) \cap \mathfrak{H}^n : B\psi \in L^2(\mathbb{R}^{3(n-1)}), A\psi \in L^2(\mathbb{R}^{3(n-1)})\Big\}$$

A model on Fock space

The Hamiltonian is defined by

$$(H\psi)^{(n)} = (-\Delta_0^* + nE_0)\psi^{(n)} + A\psi^{(n+1)}, \qquad n \geq 1$$

on the domain

$$D_{\mathrm{IBC}} = \left\{\Psi \in \mathfrak{F} : \psi^{(n)} \in D^{(n)}, A\Psi \in \mathfrak{F}, H\Psi \in \mathfrak{F}, B\psi^{(n)} = \psi^{(n-1)}\right\}.$$

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Theorem

For all $E_0 \in \mathbb{R}$ the operator (H, D_{IBC}) is essentially self adjoint and if $E_0 \geq 0$ it is bounded below.
For $E_0 > 0$ the operator is self adjoint on D_{IBC} and equals

$$H = [\mathrm{d}\Gamma(-\Delta + E_0) + a(\delta_0) + a^*(\delta_0)]_{\mathrm{ren}} + \sqrt{E_0}/4\pi.$$

The operator $[\mathrm{d}\Gamma(-\Delta + E_0) + a(\delta_0) + a^*(\delta_0)]_{\mathrm{ren}}$ is constructed using a renormalisation procedure, and unitarily equivalent to the free Hamiltonian $\mathrm{d}\Gamma(-\Delta + E_0)$ (Derezinski '03).

A model on Fock space

For $E_0 > 0$ the operator (H, D_{IBC}) is an explicit representation of $[\mathrm{d}\Gamma(-\Delta + E_0) + a(\delta_0) + a^*(\delta_0)]_{\mathrm{ren}}$.

- In the sense of distributions we have for $\psi \in D_{\mathrm{IBC}}$:

$$(H\psi)^{(n)} = (-\Delta + nE_0)\psi^{(n)} + (a^*(\delta_0)\psi)^{(n)} + A\psi^{(n+1)}.$$

- We see that

$$D_{\mathrm{IBC}} \cap \mathrm{Dom}\,(\mathrm{d}\Gamma(-\Delta + E_0)) = \{0\}$$

  This is also known for the Fröhlich Polaron (Griesemer, Wünsch '16).

A moving source in $d = 2$ dimensions

Construct a model in $d = 2$ two space dimensions on $L^2(\mathbb{R}^2) \otimes \mathfrak{F}$ with a dynamical "source" particle at position y and (singular) boundary conditions on the set $\mathscr{C}^k = \{\prod_{j=1}^k |y - x_j| = 0\}$.
For a number k of x-particles and one source let

$$\Delta_0 = \Big(\Delta, H_0^2(\mathbb{R}^{2k+2} \setminus \mathscr{C}^k)\Big)$$

$$(B\psi)(y, x_1, \dots, x_{k-1}) = 4\pi\sqrt{k} \lim_{|y - x_k| \to 0} \log|y - x_k| \psi(y, x_1, \dots, x_k)$$

$$(A\psi)(y, x_1, \dots, x_{k-1}) = \sqrt{k} \lim_{|y - x_k| \to 0} (\psi - \log|y - x_k| B\psi/(4\pi))$$

and

$$D^{(k)} = \{\psi \in D(\Delta_0^*) \cap L^2(\mathbb{R}^2) \otimes \mathfrak{H}^k : B\psi \in L^2(\mathbb{R}^{2k}), A\psi \in L^2(\mathbb{R}^{2k})\}$$

A moving source in $d = 2$ dimensions

The operator with at most N particles:

$$(H_N\psi)^{(k)} = \begin{cases} 0 & k > N \\ -\Delta_0^*\psi^{(N)} & k = N \\ -\Delta_0^*\psi^{(k)} + A\psi^{(k+1)} & k < N \end{cases}$$

with domain

$$D_N = \{\psi \in L^2(\mathbb{R}^2) \otimes \mathfrak{F} : \psi^{(k)} \in D^{(k)} \text{ and } B\psi^{(k)} = \psi^{(k-1)} \text{ for } k \leq N\}.$$

Proposition

The operator H_N is self adjoint on D_N and bounded below.

A moving source in $d = 2$ dimensions

The main ingredient of the proof is the parametrisation of $D^{(N)}$:

$$\psi^{(N)} = \varphi^{(N)} + \Gamma_N(\lambda)(B\psi^{(N)})$$

with $\varphi^{(N)} \in H^2(\mathbb{R}^{2N+2})$, $\mathrm{ran}(\Gamma_N(\lambda)) \subset \ker(-\Delta_0^* + \lambda^2)$.

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With this we construct the resolvent by solving the triangular system $(H_N + \lambda^2)\psi = g$. This is possible because $T_n := A\Gamma_{n+1}$ is bounded $D^{(n)} \to L^2 \otimes \mathfrak{H}^n$ and small compared to $H_{N-1} + \lambda^2$.

In $d = 3$ dimensions the analogue of T, the Skornyakov–Ter-Matirosyan operator, is bounded on H^1 but not on $D^{(n)}$. The proof only works for $N = 1$ (Thomas '84).

A moving source in $d = 2$ dimensions

Proposition

The limit $\lim_{N\to\infty} H_N$ exists in the strong resolvent sense and defines a self-adjoint operator H.

This is proved using that $H_M - H_N$ vanishes on all sectors with less than $\min\{M, N\}$ particles.

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