Examples of particle creation at point sources via boundary conditions

Jonas Lampart

CNRS & ICB, Université de Bourgogne Franche-Comté

August 22, 2017

joint work with J. Schmidt, S. Teufel and R. Tumulka (Tübingen)
The minimal example

A simple model for a particle that can be emitted and absorbed by a source at \( x_0 = 0 \in \mathbb{R}^3 \) (Yafaev ’92, Thomas ’84).

On \( L^2(\mathbb{R}^3) \oplus \mathbb{C} \) consider the operator

\[
H = \begin{pmatrix}
-\Delta_0^* & 0 \\
A & 0
\end{pmatrix}
\]

where

- \( \Delta_0^* \) is the adjoint of \( \Delta_0 := (\Delta, H_0^2(\mathbb{R}^3 \setminus \{0\})) \)
- \( A : D(\Delta_0^*) \to \mathbb{C} \) extends the evaluation at \( x = 0 \):

\[
A\psi = \lim_{r \to 0} \partial_r r \psi(r \omega)
\]

on the domain \( D(\Delta_0^*) \oplus \mathbb{C} \subset L^2(\mathbb{R}^3) \oplus \mathbb{C} \).

This operator is not symmetric.
It is well known that

\[
D(\Delta_0^*) = H^2(\mathbb{R}^3) \oplus \text{span}(f_\gamma)
\]

\[
f_\gamma(x) = -\frac{e^{-\gamma|x|}}{4\pi |x|}, \quad \text{Re}(\gamma) > 0.
\]
It is well known that

\[ D(\Delta_0^*) = H^2(\mathbb{R}^3) \oplus \text{span}(f_\gamma) \]

\[ f_\gamma(x) = -\frac{e^{-\gamma|x|}}{4\pi|x|}, \quad \text{Re}(\gamma) > 0. \]

Let

\[ B : D(\Delta_0^*) \to \mathbb{C}, \quad \psi \mapsto -4\pi \lim_{|x| \to 0} |x| \psi(x), \]

integration by parts shows that

\[
\langle \varphi, -\Delta_0^* \psi \rangle - \langle -\Delta_0^* \varphi, \psi \rangle \\
= \int_0^\infty \int_{S^2} \left( \left( \partial_r^2 r \bar{\varphi}(r\omega) \right) r \psi(r\omega) - r \bar{\varphi}(r\omega) \partial_r^2 r \psi(r\omega) \right) drd\omega \\
= -\langle A\varphi, B\psi \rangle + \langle B\varphi, A\psi \rangle.
\]
The minimal example

By

\[ \langle H \Phi, \Psi \rangle - \langle \Phi, H \Psi \rangle = -\langle A \varphi^{(1)}, B \psi^{(1)} \rangle + \langle B \varphi^{(1)}, A \psi^{(1)} \rangle + \langle A \varphi^{(1)}, \psi^{(0)} \rangle - \langle \varphi^{(0)}, A \psi^{(1)} \rangle \]

*H* is symmetric on the domain

\[ D_{IBC} = \left\{ \Psi = (\psi^{(1)}, \psi^{(0)}) \in D(\Delta_0^*) \oplus \mathbb{C} : B \psi^{(1)} = \psi^{(0)} \right\}. \]

The condition \( B \psi^{(1)} = \psi^{(0)} \) is a (co-dimension three) boundary condition at \( x = 0 \), we call this an *interior boundary condition* (IBC).
The minimal example

Proposition (Yafaev ’92)

The operator $H$ is self adjoint on the domain $D_{IBC}$ and $H \geq 0$.

Proof.

Since $H$ is symmetric on $D_{IBC}$ it is enough to show that $(H + \lambda^2)\psi = g$ has a unique solution $\psi \in D_{IBC}$ for $\lambda > 0$.

On the one-particle sector $\psi^{(1)} = \varphi + a f_\lambda$, with $\varphi \in H^2(\mathbb{R}^3)$ and $B\psi^{(1)} = a = \psi^{(0)}$. Then

$$(-\Delta_0^* + \lambda^2)\psi^{(1)} = (-\Delta_0 + \lambda^2)\varphi$$

After solving $(H + \lambda^2)\psi = g$ for $\varphi$, we have the equation for $\psi^{(0)}$

$$\lambda^2\psi^{(0)} + A f_\lambda \psi^{(0)} = g^{(0)} - A(-\Delta_0 + \lambda^2)^{-1}g^{(1)} = \lambda$$

which is solvable for $\lambda > 0$. 
An arbitrary number of particles can be created/annihilated by a source at the origin. 
Let $\mathcal{F}$ be the bosonic Fock space over $L^2(\mathbb{R}^3)$ and $\mathcal{F}^n$ its $n$-particle sector. The singular set in the configuration space of $n$-particles is the set $\mathcal{C}^n$ with at least one particle at the origin.
A model on Fock space

An arbitrary number of particles can be created/annihilated by a source at the origin.

Let $\mathcal{F}$ be the bosonic Fock space over $L^2(\mathbb{R}^3)$ and $\mathcal{H}^n$ its $n$-particle sector. The singular set in the configuration space of $n$-particles is the set $\mathcal{C}^n$ with at least one particle at the origin. For $n$ particles let

$$\Delta_0 = \left( \Delta, H^2_0(\mathbb{R}^{3n} \setminus \mathcal{C}^n) \right)$$

$$(B\psi)(x_1, \ldots, x_{n-1}) = -4\pi \sqrt{n} \lim_{|x_n| \to 0} |x_n| \psi(x_1, \ldots, x_n)$$

$$(A\psi)(x_1, \ldots, x_{n-1}) = \sqrt{n} \lim_{r \to 0} \partial_r r \psi(x_1, \ldots, x_{n-1}, r\omega)$$

and

$$D^{(n)} := \left\{ \psi \in D(\Delta_0^*) \cap \mathcal{H}^n : B\psi \in L^2(\mathbb{R}^{3(n-1)}), \ A\psi \in L^2(\mathbb{R}^{3(n-1)}) \right\}$$
The Hamiltonian is defined by

\[ (H\psi)^{(n)} = (-\Delta^* + nE_0)\psi^{(n)} + A\psi^{(n+1)}, \quad n \geq 1 \]

on the domain

\[ D_{\text{IBC}} = \left\{ \Psi \in \mathcal{F} : \psi^{(n)} \in D^{(n)}, A\Psi \in \mathcal{F}, H\Psi \in \mathcal{F}, B\psi^{(n)} = \psi^{(n-1)} \right\}. \]
The Hamiltonian is defined by

\[(H\psi)^{(n)} = (-\Delta^* + nE_0)\psi^{(n)} + A\psi^{(n+1)}, \quad n \geq 1\]

on the domain

\[D_{IBC} = \left\{ \Psi \in \mathcal{F} : \psi^{(n)} \in D^{(n)}, A\Psi \in \mathcal{F}, H\Psi \in \mathcal{F}, B\psi^{(n)} = \psi^{(n-1)} \right\}.\]

**Theorem**

*For all* \(E_0 \in \mathbb{R}\) *the operator* \((H, D_{IBC})\) *is essentially self adjoint and if* \(E_0 \geq 0\) *it is bounded below.*

*For* \(E_0 > 0\) *the operator is self adjoint on* \(D_{IBC}\) *and equals*

\[H = \left[ d\Gamma(-\Delta + E_0) + a(\delta_0) + a^*(\delta_0) \right]_{\text{ren}} + \sqrt{E_0}/4\pi.\]

The operator \(\left[ d\Gamma(-\Delta + E_0) + a(\delta_0) + a^*(\delta_0) \right]_{\text{ren}}\) *is constructed using a renormalisation procedure, and unitarily equivalent to the free Hamiltonian* \(d\Gamma(-\Delta + E_0)\) *(Derezinski '03).*
For $E_0 > 0$ the operator $(H, D_{IBC})$ is an explicit representation of $[d\Gamma(-\Delta + E_0) + a(\delta_0) + a^*(\delta_0)]_{\text{ren}}$.

- In the sense of distributions we have for $\psi \in D_{IBC}$:

$$
(H\psi)^{(n)} = (-\Delta + nE_0)\psi^{(n)} + (a^*(\delta_0)\psi)^{(n)} + A\psi^{(n+1)}.
$$

- We see that

$$
D_{IBC} \cap \text{Dom} \left( d\Gamma(-\Delta + E_0) \right) = \{0\}
$$

This is also known for the Fröhlich Polaron (Griesemer, Wünsch '16).
Construct a model in $d = 2$ two space dimensions on $L^2(\mathbb{R}^2) \otimes \mathcal{F}$ with a dynamical “source” particle at position $y$ and (singular) boundary conditions on the set $\mathcal{C}^k = \{\prod_{j=1}^k |y - x_j| = 0\}$. For a number $k$ of $x$-particles and one source let

$$\Delta_0 = \left(\Delta, H_0^2(\mathbb{R}^{2k+2} \setminus \mathcal{C}^k)\right)$$

$$(B\psi)(y, x_1, \ldots, x_{k-1}) = 4\pi\sqrt{k} \lim_{|y - x_k| \to 0} \log |y - x_k| \psi(y, x_1, \ldots, x_k)$$

$$(A\psi)(y, x_1, \ldots, x_{k-1}) = \sqrt{k} \lim_{|y - x_k| \to 0} (\psi - \log |y - x_k| B\psi / (4\pi))$$

and

$$D^{(k)} = \{\psi \in D(\Delta^*_0) \cap L^2(\mathbb{R}^2) \otimes \mathcal{F}^k : B\psi \in L^2(\mathbb{R}^{2k}), A\psi \in L^2(\mathbb{R}^{2k})\}$$
The operator with at most \( N \) particles:

\[
(H_N \psi)^{(k)} = \begin{cases} 
  0 & k > N \\
  - \Delta_0^* \psi^{(N)} & k = N \\
  - \Delta_0^* \psi^{(k)} + A \psi^{(k+1)} & k < N 
\end{cases}
\]

with domain

\[
D_N = \{ \psi \in L^2(\mathbb{R}^2) \otimes \mathcal{F} : \psi^{(k)} \in D^{(k)} \text{ and } B \psi^{(k)} = \psi^{(k-1)} \text{ for } k \leq N \}.
\]

**Proposition**

The operator \( H_N \) is self adjoint on \( D_N \) and bounded below.
The main ingredient of the proof is the parametrisation of $D^{(N)}$:

$$\psi^{(N)} = \varphi^{(N)} + \Gamma_N(\lambda)(B\psi^{(N)})$$

with $\varphi^{(N)} \in H^2(\mathbb{R}^{2N+2})$, $\text{ran}(\Gamma_N(\lambda)) \subset \ker(-\Delta_0^* + \lambda^2)$. 

With this we construct the resolvent by solving the triangular system $(H_N + \lambda^2)\psi = g$. This is possible because $T_n := A\Gamma_n + 1$ is bounded $D(n) \to L^2 \otimes H_n$ and small compared to $H_N - 1 + \lambda^2$.

In $d = 3$ dimensions the analogue of $T$, the Skornyakov–Ter-Matirosyan operator, is bounded on $H^1$ but not on $D(n)$. The proof only works for $N = 1$ (Thomas '84).
The main ingredient of the proof is the parametrisation of $D^{(N)}$:

$$\psi^{(N)} = \varphi^{(N)} + \Gamma_N(\lambda)(B\psi^{(N)})$$

with $\varphi^{(N)} \in H^2(\mathbb{R}^{2N+2})$, $\text{ran}(\Gamma_N(\lambda)) \subset \ker(-\Delta_0^{*} + \lambda^2)$.

With this we construct the resolvent by solving the triangular system

$$(H_N + \lambda^2)\psi = g.$$  This is possible because $T_n := A\Gamma_{n+1}$ is bounded $D^{(n)} \to L^2 \otimes \mathcal{H}^n$ and small compared to $H_{N-1} + \lambda^2$.

In $d = 3$ dimensions the analogue of $T$, the Skornyakov–Ter-Matirosyan operator, is bounded on $H^1$ but not on $D^{(n)}$. The proof only works for $N = 1$ (Thomas '84).
Proposition

The limit $\lim_{N \to \infty} H_N$ exists in the strong resolvent sense and defines a self-adjoint operator $H$.

This is proved using that $H_M - H_N$ vanishes on all sectors with less than $\min\{M, N\}$ particles.


J.L., J. Schmidt: Particle creation by boundary conditions at moving sources in one and two dimensions. *In preparation.*