Interacting electrons in a random background

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The \( n \) particle system

- On \( \Lambda \) large cube of \( \mathbb{R}^d \), consider \( H_\omega (\Lambda) \) a random Schrödinger operator (single particle model).
- On \( \bigwedge_{j=1}^n L^2(\Lambda) = L^2_\nu (\Lambda^n) \), consider the free operator

\[
H^0_\omega (\Lambda, n) = \sum_{i=1}^n \underbrace{1 \otimes \cdots \otimes 1}_{i-1 \text{ times}} \otimes H_\omega (\Lambda) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i \text{ times}}.
\]

- Pick \( U : \mathbb{R} \rightarrow \mathbb{R}^+ \) pair interaction potential
  
Define

\[
H^U_\omega (\Lambda, n) = H^0_\omega (\Lambda, n) + W_n, \quad \text{where} \quad W_n (x^1, \ldots, x^n) := \sum_{i<j} U (x^i - x^j).
\]

Thermodynamic limit

- Let \( E^U_\omega (\Lambda, n) \) be the ground state energy of \( H^U_\omega (\Lambda, n) \).
- Let \( \Psi^U_\omega (\Lambda, n) \) be the associated eigenfunction.

Problem

Describe \( E^U_\omega (\Lambda, n) \) and \( \Psi^U_\omega (\Lambda, n) \) in the limit \( |\Lambda| \rightarrow +\infty \) and \( \frac{n}{|\Lambda|} \rightarrow \rho > 0 \).
Description of the ground state: the (reduced) density matrices:

Let $\Psi \in L^2_\Lambda(A^n)$ be a normalized $n$-electron wave function.

- $k$-particle density matrix: $\gamma^{(k)}_{\Psi}(x,y) = \binom{n}{k} \int_{\Lambda^{n-1}} \Psi(x,\tilde{x})\Psi^*(y,\tilde{x})d\tilde{x}.$
- $\gamma^{(k)}_{\Psi}$ non negative trace class operator satisfying $\text{tr}\gamma^{(k)}_{\Psi} = \binom{n}{k}.$

The non interacting system Let $(E^p_\rho)_{p \geq 1}$ (resp. $(\psi^p_\rho)_{p \geq 1}$) be the eigenvalues (resp. associated eigenfunctions) of $H_\omega(\Lambda)$.

Define the counting fct per volume unit: $N_{H_\omega(\Lambda)}(E) = \frac{\#\{\text{e.v. of } H_\omega(\Lambda) \text{ in } (-\infty,E]\}}{|\Lambda|}$.

As $N_{H_\omega(\Lambda)}(E_n) = n/|\Lambda| \rightarrow \rho$, one has

$$\frac{E^0_{\omega}(\Lambda,n)}{n} = \frac{1}{n} \sum_{j=1}^{n} E_j = \frac{|\Lambda|}{n} \int_{-\infty}^{E_n} E dN_{H_\omega(\Lambda)}(E) \rightarrow \frac{1}{\rho} \int_{-\infty}^{E_\rho} E dN(E)$$

where $N(E_\rho) = \rho; E_\rho = $ Fermi energy and $N(E) := \lim_{|\Lambda| \rightarrow +\infty} N_{H_\omega(\Lambda)}(E)$ (IDS of $H_\omega$).

Moreover, for non interacting ground state

$$\gamma_{\Psi^0_\omega(\Lambda,n)}^{(1)} = \sum_{k=0}^{n} |\psi_k^{\omega_0}\rangle \langle \psi_k^{\omega_0}| \rightarrow \mathbb{1}_{(-\infty,E_\rho]}(H_\omega).$$

A simple one-dimensional random model

The pieces (or Luttinger-Sy) model

- On $\mathbb{R}$, consider Poisson process $d\mu(\omega)$ of intensity $\mu$ i.e.

  $$d\mu(\omega) = \sum_{k \in \mathbb{Z}} \delta_{\Delta_k}(\omega).$$

  - For $\Lambda = [-L/2,L/2]$, on $L^2(\Lambda)$, define

  $$H_\omega(L) = \bigoplus_{k \in \mathbb{Z}} -\frac{d^2}{dx^2} \bigg|_{\Delta_k \cap \Lambda}$$

  where $\Delta_k = \Delta_k(\omega) = [x_k,x_{k+1}]$.

  - Integrated density of states

  $$N(E) := \lim_{L \rightarrow +\infty} \frac{\#\{\text{eigenvalues of } H_\omega(L) \text{ in } (-\infty,E]\}}{L}$$

  $$= \frac{\exp(-\ell_E)}{1 - \exp(-\ell_E)} 1_{E \geq 0} \text{ where } \ell_E := \frac{\pi}{\sqrt{E}}.$$
The \( n \) particle system

- On \( \bigwedge_{i=1}^n L^2([-L/2,L/2]) = L^2([-L/2,L/2]^n) \), consider the free operator
  \[
  H_{\omega}^0(L,n) = \sum_{i=1}^n 1 \otimes \cdots \otimes 1 \otimes H_{\omega}(L) \otimes 1 \otimes \cdots \otimes 1. 
  \]

- Pick \( U : \mathbb{R} \to \mathbb{R}^+ \) not identically vanishing, even, bounded.

Define

\[
H_{\omega}^U(L,n) = H_{\omega}^0(L,n) + W_n, \quad \text{where} \quad W_n(x^1, \cdots, x^n) := \sum_{i<j} U(x_i - x_j). 
\]

The non interacting system : the ground state energy per particle

\[
E_0(\rho) = \lim_{L \to +\infty, n/L \to \rho} \frac{E_{\omega}^0(L,n)}{n} = E_{\rho} \left( 1 + O\left( \sqrt{E_{\rho}} \right) \right) = \pi^2 |\log \rho|^{-2} \left( 1 + O\left( |\log \rho|^{-1} \right) \right). 
\]
The 2 point correlation function

**Theorem**

Assume $U$ cpt support. There exists $\rho_U > 0$ and $\mu > 0$ s.t. for $0 < \rho < \rho_U$ sufficiently small, one has

$$\sup_{\Lambda} \sup_{(x,x',y,y') \in \Lambda^4} \mathbb{E}(|\gamma^{(2)}_{\omega}(L,n)(x,y;x',y')|e^{\mu(|x-y| + |x'-y'|)}) < +\infty.$$ 

Ground state energy per particle

**Theorem (K.-Veniaminov)**

Under our assumptions on $U$, $\omega$—almost surely, the following limit exists, is independent of $\omega$ and admits the asymptotic expansion

$$\mathcal{E}^U(\rho) : = \lim_{L \to +\infty} \frac{E^U(\omega,L,n)}{n} = \mathcal{E}^0(\rho) + \frac{\pi^2 \gamma_*}{|\log \rho|^3} \rho + o \left( \frac{\rho}{|\log \rho|^3} \right),$$

where $\gamma_* = 1 - \exp \left( -\frac{\gamma}{8\pi^2} \right).$

Systems of two electrons within the same piece :

**Lemma (K.-Veniaminov)**

Assume that $U \in L^p(\mathbb{R})$ for some $p \in (1, +\infty]$ and that $\int \mathbb{R} x^2 U(x)dx < +\infty$. Consider two electrons in $[0, \ell]$ interacting via the pair potential $U$, i.e., on $L^2([0, \ell]) \wedge L^2([0, \ell])$, consider the Hamiltonian

$$- \frac{d^2}{dx_1^2} - \frac{d^2}{dx_2^2} + U(x_1 - x_2). \quad (1)$$

Then, for large $\ell$, $E^{2,U}(\ell)$, its ground state energy admits the following expansion

$$E^{2,U}(\ell) = \frac{5\pi^2}{\ell^2} + \frac{\gamma}{\ell^3} + o \left( \ell^{-3} \right)$$

where $\gamma := \frac{5\pi^2}{2} \left\langle \bullet \sqrt{U(\bullet)}, \left( Id + \frac{1}{2} U^{1/2} (-\Delta_1)^{-1} U^{1/2} \right)^{-1} \bullet \sqrt{U(\bullet)} \right\rangle.$

Uniqueness of the ground state :

**Theorem (K.-Veniaminov)**

Assume $U$ is analytic. Then, for any $L$ and $n$, $H^U_{\omega}(L,n)$ has a unique ground state $\omega$—almost surely.
Interacting electrons in a random background

There exists a ground state of the system. In particular, we assume that the support of the interacting potential is compact. Theorem (K.-Veniaminov)

We assume $U$ is compact support. There exists $\rho_0 > 0$ s.t. for $\rho \in (0, \rho_0)$, $\omega$-a.s., one has

\[
\limsup_{L \to +\infty} \frac{1}{n} \left\| \sum_{\ell \rho \leq \gamma \leq \Delta_k \leq 2\ell \rho \log (1-\gamma)} \gamma^{(1)}_{\psi_{\omega}^{\text{opt}}}(L, n) \right\|_1 \lesssim \frac{\rho}{|\log \rho|}.
\]

Quantification of the influence of interactions

Influence of interactions on the ground state is essentially described by

\[
\gamma^{(1)}_{\psi^{\text{opt}}_{\omega}}(L, n) - \gamma^{(1)}_{\psi^{(1)}_{\omega}}(L, n) = \sum_{\ell \rho \leq \gamma \leq \Delta_k \leq 2\ell \rho \log (1-\gamma)} \left( \sum_{2\ell \rho \leq \Delta_k \leq 2\ell \rho - \log (1-\gamma)} \gamma^{(1)}_{\varphi^{(2)}_{\Delta_k}} - \gamma^{(1)}_{\varphi^{(2)}_{\Delta_k}} \right)
\]

In particular,

\[
\lim_{L \to +\infty} \frac{1}{n} \left\| \gamma^{(1)}_{\psi^{\text{opt}}_{\omega}}(L, n) - \gamma^{(1)}_{\psi^{(1)}_{\omega}}(L, n) \right\|_1 = 2\gamma_s \rho + O\left(\frac{\rho}{|\log \rho|}\right),
\]

and

\[
\limsup_{L \to +\infty} \frac{1}{n} \left\| \gamma^{(2)}_{\psi^{\text{opt}}_{\omega}}(L, n) - \gamma^{(2)}_{\psi^{(2)}_{\omega}}(L, n) \right\|_1 \lesssim \frac{\rho}{|\log \rho|}.
\]
A more realistic random model

The Poisson potential

- On $\mathbb{R}$, consider the Poisson point process $d\mu(\omega)$ of intensity 1
  
  $$d\mu(\omega) = \sum_{k \in \mathbb{Z}} \delta_{x_k}(\omega)$$

- Fix $v : \mathbb{R} \rightarrow \mathbb{R}^+$, $v \geq 0$, continuous compactly supported and define
  
  $$H_\omega = -\Delta + \int_{\mathbb{R}} v(x-y)d\mu(y) = -\Delta + \sum_{k \in \mathbb{Z}} v(x-x_k(\omega)).$$

- Integrated density of states cannot be computed explicitly

**Theorem**

For $E$ small, one has the asymptotic expansion

$$-\log N(E) = \ell_E - c_0 + \sum_{k \geq 1} c_k \ell_E^{-k} \quad \text{where} \quad \ell_E = \frac{\pi}{\sqrt{E}}.$$

Ground state energy per particle

**Theorem**

For $U$ repulsive and continuous of cpct support, $\omega$—almost surely, the following limit exists, is independent of $\omega$ and admits the asymptotic expansion

$$\mathcal{E}^U(\rho) := \lim_{L \to +\infty \atop n/L \to \rho} \frac{E_{\omega}^U(L,n)}{n} = \mathcal{E}^0(\rho) + \frac{\pi^2 \gamma}{|\log \rho|^3} \rho + o \left( \frac{\rho}{|\log \rho|^3} \right).$$

One also has description of ground states (not necessarily unique).

Some open questions

- Prove exponential decay of 2 point correlation for random Schrödinger model.
  (related work of Mastropietro (2014-16)).

- Study the influence of the range of the interaction $U$.

- What happens if $\rho$ is increased ?

- What happens for random Schrödinger model in dimension $d \geq 2$ ?

- What happens when $T \neq 0$ ?
  (Mastropietro (2014-16)).