Manifestations of dynamical localization in the random XXZ spin chain

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Many-body localization in the droplet spectrum of the random XXZ..., arXiv:1703.07483
Manifestations of dynamical localization in the disordered XXZ spin...., arXiv:1708.00474

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The infinite XXZ chain in a random field is given by the Hamiltonian

$$H_\omega = \sum_{i\in\mathbb{Z}} \left\{ \tfrac{1}{4}\left(I - \sigma_i^z \sigma_{i+1}^z\right) - \tfrac{1}{4\Delta}\left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right)\right\} + \lambda \sum_{i\in\mathbb{Z}} \omega_i \mathcal{N}_i,$$

acting on $\bigotimes_{i\in\mathbb{Z}} \mathbb{C}_i^2$, $\mathbb{C}_i^2 = \mathbb{C}^2$ for all i, where

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We have $\quad \sigma(H_\omega) = \{0\} \cup \left[1 - \frac{1}{\Delta}, \infty\right) \quad$ almost surely.

XXZ chain Hamiltonian in finite intervals

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- Unique ground state $\psi_0 = \psi_0^{(L)}$ determined by $\mathcal{N}_i \psi_0 = 0$ for all i.
- The spectrum of $H^{(L)} = H_\omega^{(L)}$ is almost surely simple, so that its normalized eigenvectors can be labeled as ψ_E, $E \in \sigma(H^{(L)})$.

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We will say that we have droplet localization in an interval I if the conclusions of the theorem hold in the interval I.

Best possible interval for droplet localization

We proved droplet localization on intervals

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- If $\ell \geq 1$, we set $\mathcal{S}_{X,\ell} = (\mathcal{S}_X)_\ell = [s_X - \ell, r_X + \ell] \cap [-L, L]$.

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♦ A local observable X with support $J \subset [-L, L]$ is an operator on $\otimes_{j \in J} \mathbb{C}_j^2$, considered as an operator on $\mathcal{H}^{(L)}$ by acting as the identity on spins not in J. We always take J to an interval. Supports of observables are not uniquely defined.

♦ Given a local observable X, we will generally specify a support for X, denoted by $\mathcal{S}_X = [s_X, r_X]$. We always assume $\emptyset \neq \mathcal{S}_X \subset [-L, L]$.

♦ If $\ell \geq 1$, we set $\mathcal{S}_{X,\ell} = (\mathcal{S}_X)_\ell = [s_X - \ell, r_X + \ell] \cap [-L, L]$.

♦ Given two local observables X, Y we set $\operatorname{dist}(X, Y) = \operatorname{dist}(\mathcal{S}_X, \mathcal{S}_Y)$.

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The time evolution of a local observable X under $H^{(L)}$ is given by

$$\tau_t(X) = \tau_t^{(L)}(X) = \mathrm{e}^{itH^{(L)}} X \mathrm{e}^{-itH^{(L)}} \quad \text{for} \quad t \in \mathbb{R}.$$

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Thus, given an energy interval J, we consider the sub-Hilbert space $\operatorname{Ran} P_J^{(L)}$, spanned by the the eigenstates of $H^{(L)}$ with energies in J, and localize an observable X in the energy interval J by considering its restriction to $\operatorname{Ran} P_J^{(L)}$,

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Clearly
$$\tau_t(X_J) = (\tau_t(X))_J \quad .$$

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$X_I = \left(X_{I_0}\right)_I \quad \Longrightarrow \quad$ the theorem holds with I substituted for I_0.

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$$\begin{aligned} &\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left[\tau_t\left(X_{I_0}\right), Y_{I_0}\right] - \left(\tau_t\left(X\right)P_0 Y - Y P_0 \tau_t\left(X\right)\right)_I \right\|_1\right) \\ &\qquad\qquad \leq C\|X\|\|Y\|\mathrm{e}^{-\frac{1}{8}m\,\mathrm{dist}(X,Y)}, \end{aligned} \tag{1}$$

$$\begin{aligned} &\mathbb{E}\left(\sup_{t,s\in\mathbb{R}} \left\| \left[\left[\tau_t\left(X_{I_0}\right), \tau_s\left(Y_{I_0}\right)\right], Z_{I_0}\right] \right\|_1\right) \\ &\qquad\qquad \leq C\|X\|\|Y\|\|Z\|\mathrm{e}^{-\frac{1}{8}m\,\min\{\mathrm{dist}(X,Y),\mathrm{dist}(X,Z),\mathrm{dist}(Y,Z)\}}. \end{aligned}$$

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Moreover, the estimate (1) is not true without the counterterms.

Correlators

We define the truncated time evolution of an observable X in the energy window I by $(H = H_\omega^{(L)})$,

$$\tau_t^I(X) = \mathrm{e}^{itH_I} X \mathrm{e}^{-itH_I}, \quad \text{where} \quad H_I = HP_I.$$

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The correlator operator of two observables X and Y in the energy window I is given by ($\bar{P}_I = 1 - P_I$)

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We are interested in quantities of the form ($K \subset I$)

$$R_K(\tau_t^I(X), Y) = \left(\tau_t^I(X)\, Y\right)_K - \left(\tau_t^I(X)\right)_K Y_K = \left(\tau_t^I(X)\, Y\right)_K - \tau_t(X_K)\, Y_K.$$

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While it is obvious where the first counterterm comes from, the same is not true of the second, where the time evolution seems to sit in the *wrong* place: it is $\tau_t^K(Y)$ and not $\tau_t^K(X)$. It turns out this term encodes information about the states above the energy window K, and the appearance of $\tau_t^K(Y)$ is related to the reduction of this data to P_0.

Decomposition of local observables

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Given a local observable X, we define projections $P_{\pm}^{(X)}$ by

$$P_{+}^{(X)} = \bigotimes_{j \in \mathcal{S}_X} \tfrac{1}{2}(1 + \sigma_j^z) \quad \text{and} \quad P_{-}^{(X)} = 1 - P_{+}^{(S)}.$$

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Moreover, since $P_{+}^{(X)}$ is a rank one projection on $\mathcal{H}_{\mathcal{S}_X}$, we must have

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In particular,

$$(X - \zeta_X)^{+,+} = 0 \quad \text{and} \quad \|X - \zeta_X\| \leq 2 \|X\|,$$

so we can assume $X^{+,+} = 0$ in the proofs.

Consequences of droplet localization

Lemma

Let X, Y be local observables, $\ell \geq 1$. Then

$$\mathbb{E}\left(\sup_{g\in G_{I_0}} \left\| P_-^{(X)} g(H) P_-^{(Y)} \right\|_1\right) \leq C\mathrm{e}^{-m\,\mathrm{dist}(X,Y)}$$

$$\mathbb{E}\left(\left\| P_-^{(Y)} P_-^{(X)} P_{I_0} \right\|_1\right) \leq C\mathrm{e}^{-\frac{1}{2}m\,\mathrm{dist}(X,Y)}$$

$$\mathbb{E}\left(\sup_{I\in G_I} \left\| P_-^{(X)} g(H) P_+^{(\mathcal{S}_{X,\ell})} \right\|_1\right) \leq C\mathrm{e}^{-m\ell}$$

$$\mathbb{E}\left(\sup_{g\in G_I} \left\| P_+^{(\mathcal{S}_{Y,\ell}^c)} g(H) P_+^{(\mathcal{S}_{X,\ell}^c)} \right\|_1\right) \leq C\mathrm{e}^{-m(\mathrm{dist}(X,Y)-2\ell)}$$

Non-spreading of information- Sketch of proof

To prove: Given a local observables X, $t \in \mathbb{R}$ and $\ell > 0$, there is a local observable $X_\ell(t) = (X_\ell(t))_\omega$ with support $\mathcal{S}_{X,\ell}$ satisfying

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Sketch of proof: Let $\mathcal{S}_X = [s_X, r_X]$, recall $\mathcal{S}_{X,\ell} = [s_X - \ell, r_X + \ell]$, and set

$$\mathcal{O} = [-L, L] \setminus \mathcal{S}_{X,\frac{\ell}{2}} = [-L, s_X - \tfrac{\ell}{2}) \cup (r_X + \tfrac{\ell}{2}, L]$$
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We first prove that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left(P_+^{(\mathcal{O})}\tau_t\left(X_{I_0}\right)P_+^{(\mathcal{O})} - \tau_t(X)\right)_{I_0} \right\|_1\right) \leq C\|X\|\mathrm{e}^{-\frac{1}{16}m\ell}.$$

We now observe that for all observables Z we have

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We conclude that

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Since $P_+^{(\mathcal{O})} \widetilde{\tau_t\left(X_{I_0}\right)}$ does not have support in $\mathcal{S}_{X,\ell}$, we now define

$$X_\ell(t) = P_+^{(\mathcal{T})} \widetilde{\tau_t\left(X_{I_0}\right)} \quad \text{for} \quad t \in \mathbb{R},$$

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where $\tilde{Z}$ is an observable with $\mathcal{S}_{\tilde{Z}} = \mathcal{S}_{X,\frac{\ell}{2}}$ and $\|\tilde{Z}\| \le \|Z\|$.
We conclude that

$$\mathbb{E}\left(\sup_{t\in\mathbb{R}} \left\| \left(P_+^{(\mathcal{O})} \widetilde{\tau_t(X_{I_0})} - \tau_t(X) \right)_{I_0} \right\|_1 \right) \le C \|X\| \mathrm{e}^{-\frac{1}{16} m\ell}.$$

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The commutator can be estimated by the Lieb-Robinson bound.

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For $E, E' \in K$ we have

$$P_E \left(\int_{\mathbb{R}} \mathrm{e}^{-irH} Y \tau_r(X) \hat{f}(r)\, \mathrm{d}r \right) P_{E'} = P_E Y f(E + E' - H) X P_{E'}$$
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$$0 \leq h \leq 1, \ \operatorname{supp} h \subset (-\varepsilon, \varepsilon), \ h(0) = 1, \ \text{and} \ \left|\hat{h}(t)\right| \leq C\mathrm{e}^{-c|t|^{\frac{1}{2}}}$$

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Let X, Y be local observables with $X^{+,+} = Y^{+,+} = 0$. The Lemmas yield

$$\begin{aligned} \|(XP_0Y)_K\| &= \|(Xh(H)Y)_K\| \\ &\le C\,\|X\|\,\|Y\|\,\mathrm{e}^{-m_1(\operatorname{dist}(X,Y))^{\frac{1}{2}}} + C' \sup_{r\in\mathbb{R}} \left\|(YP_{K_h}\tau_r(X))_K\right\|, \end{aligned}$$

where $K_h \subset [2\Theta_0 - \varepsilon, 2\Theta_2 + \varepsilon] \subset [\Theta_0, \Theta_1] = I$.

We can prove

$$\mathbb{E}\left(\sup_{r\in\mathbb{R}}\left\|\left(YP_{K_h}\tau_r\left(X\right)\right)_K\right\|\right)\leq C\left\|X\right\|\left\|Y\right\|\mathrm{e}^{-\frac{1}{8}m\,\mathrm{dist}(X,Y)},$$

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In particular, it follows that we have, uniformly in L,

$$\mathbb{E}\left(\left\|\left(\sigma_i^x P_0^{(L)}\sigma_j^x\right)_K\right\|\right)\leq C\mathrm{e}^{-m_2\left(|i-j|\right)^{\frac{1}{2}}}\quad\text{for all}\quad i,j\in[-L,L]. \qquad (2)$$

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(2) and (3) give a contradiction $\implies$ $\Theta_1\leq 2\Theta_0$.