Manifestations of dynamical localization in the random XXZ spin chain

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Many-body localization in the droplet spectrum of the random XXZ..., arXiv:1703.07483
Manifestations of dynamical localization in the disordered XXZ spin..., arXiv:1708.00474

Quantissima in the Serenissima II
Mathematical challenges in classical & quantum statistical mechanics
Venice, 21-25 August 2017
The random XXZ quantum spin chain Hamiltonian

The infinite XXZ chain in a random field is given by the Hamiltonian

$$H_\omega = \sum_{i \in \mathbb{Z}} \left\{ \frac{1}{4} (1 - \sigma_i^z \sigma_{i+1}^z) - \frac{1}{4\Delta} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) \right\} + \lambda \sum_{i \in \mathbb{Z}} \omega_i N_i,$$

acting on $\bigotimes_{i \in \mathbb{Z}} \mathbb{C}_i^2$, $\mathbb{C}_i^2 = \mathbb{C}^2$ for all $i$, where
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5. \( \omega = \{\omega_i\}_{i \in \mathbb{Z}} \) are independent identically distributed random variables whose probability distribution \( \mu \) is absolutely continuous with a bounded density, with \( \{0, 1\} \subset \text{supp}\ \mu \subset [0, 1] \).
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\( H_\omega \) is a self-adjoint operator on an appropriately defined Hilbert space \( \mathcal{H} \).

We have \( \sigma(H_\omega) = \{0\} \cup \left[ 1 - \frac{1}{\Delta}, \infty \right) \) almost surely.
Consider the finite interval $[-L, L] = [-L, L] \cap \mathbb{Z}$, $L \in \mathbb{N}$, and set

$$H^{(L)}_{\omega} = \sum_{i=-L}^{L-1} \left\{ \frac{1}{4} (I - \sigma_i^z \sigma_{i+1}^z) - \frac{1}{4\Delta} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) \right\} + \lambda \sum_{i=-L}^{L} \omega_i N_i$$

$$+ \beta (N_{-L} + N_L)$$

on $\mathcal{H}^{(L)} = \bigotimes_{i \in [-L, L]} \mathbb{C}^2_i$.
XXZ chain Hamiltonian in finite intervals

Consider the finite interval \([-L, L] = [-L, L] \cap \mathbb{Z}, L \in \mathbb{N}\), and set

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- We fix \(\beta \geq \frac{1}{2}(1 - \frac{1}{\Delta})\), so

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- Unique ground state \(\psi_0 = \psi_0^{(L)}\) determined by \(\mathcal{N}_i \psi_0 = 0\) for all \(i\).
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- The spectrum of \( H^{(L)} = H^{(L)}_\omega \) is almost surely simple, so that its normalized eigenvectors can be labeled as \( \psi_E, \ E \in \sigma(H^{(L)}) \).
The droplet spectrum

The droplet spectrum for the free ($\lambda = 0$) XXZ spin chain is given by

$$I_1 = \left[1 - \frac{1}{\Delta}, 2(1 - \frac{1}{\Delta})\right].$$
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\[ G_I = \{ g : \mathbb{R} \to \mathbb{C} \; \text{Borel measurable,} \; |g| \leq \chi_I \}. \]
Theorem (Localization in the droplet spectrum)

There exists a constant $K > 0$ with the following property:

If $\Delta > 1$, $\lambda > 0$, and $0 < \delta < 1$ satisfy

$$\lambda \left( \delta (\Delta - 1) \right)^{1/2} \min \{1, \delta (\Delta - 1) \} \geq K,$$

there exist constants $C < \infty$ and $m > 0$ such that we have, uniformly in $\mathcal{L}$,

$$E \left( \sum_{E \in \sigma} I_{1,\delta}(H(\mathcal{L})) \parallel N_i \psi_E \parallel \parallel N_j \psi_E \parallel \right) \leq Ce^{-m|i-j|}$$

for all $i, j \in [-L, L]$.

We will say that we have droplet localization in an interval $I$ if the conclusions of the theorem hold in the interval $I$. 

Abel Klein

Dynamical localization in the disordered XXZ spin chain
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\mathbb{E} \left( \sum_{E \in \sigma_{I_1, \delta}(H(L))} \|N_i \psi_E\| \|N_j \psi_E\| \right) \leq C e^{-m|j-i|} \text{ for all } i, j \in [-L, L],
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and, as a consequence,

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Best possible interval for droplet localization

We proved droplet localization on intervals

\[ I_{1,\delta} = \left[ 1 - \frac{1}{\Delta}, (2 - \delta)(1 - \frac{1}{\Delta}) \right] \subset \left[ 1 - \frac{1}{\Delta}, 2(1 - \frac{1}{\Delta}) \right]. \]
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Droplet localization for the random XXZ spin chain (in the sense of the Theorem) is not possible outside the droplet spectrum.
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**Theorem**

Suppose we have droplet localization in the interval $I = \left[1 - \frac{1}{\Delta}, \Theta\right]$. 

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**Theorem**

Suppose we have droplet localization in the interval \( I = \left[ 1 - \frac{1}{\Delta}, \Theta \right] \). Then

\[ \Theta \leq 2 \left( 1 - \frac{1}{\Delta} \right), \]
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Suppose we have droplet localization in the interval \( l = \left[ 1 - \frac{1}{\Delta}, \Theta \right] \).

Then

\[ \Theta \leq 2\left(1 - \frac{1}{\Delta}\right), \]

that is, we must have

\[ l = l_{1,\delta} \quad \text{for some} \quad 0 \leq \delta < 1. \]
Consequences of droplet localization

Preliminaries

- $H = H_\omega$ will be a random XXZ spin chain satisfying droplet localization in the interval $I = I_{1,\delta} = \left[1 - \frac{1}{\Delta}, (2 - \delta)(1 - \frac{1}{\Delta})\right]$. 

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♦ $P_B^{(L)} = \chi_B(H^{(L)})$ for $B \subset \mathbb{R}$, with $P_E^{(L)} = P_{\{E\}}^{(L)}$ for $E \in \mathbb{R}$.
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- $I_0 = \left[0, (2 - \delta)(1 - \frac{1}{\Delta})\right] \approx \{0\} \cup I \implies P^{(L)}_{I_0} = P^{(L)}_0 + P^{(L)}_I$. 

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♦ $P_B^{(L)} = \chi_B(H^{(L)})$ for $B \subset \mathbb{R}$, with $P_E^{(L)} = P_{\{E\}}^{(L)}$ for $E \in \mathbb{R}$.

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♦ A local observable $X$ with support $J \subset [-L, L]$ is an operator on $\bigotimes_{j \in J} \mathbb{C}_j^2$, considered as an operator on $\mathcal{H}^{(L)}$ by acting as the identity on spins not in $J$. We always take $J$ to an interval. Supports of observables are not uniquely defined.
H = H_\omega \text{ will be a random XXZ spin chain satisfying droplet localization in the interval } I = I_{1,\delta} = \left[ 1 - \frac{1}{\Delta}, (2 - \delta)(1 - \frac{1}{\Delta}) \right].

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I_0 = \left[ 0, (2 - \delta)(1 - \frac{1}{\Delta}) \right] \approx \{0\} \cup I \implies P_{I_0}^{(L)} = P_0^{(L)} + P_I^{(L)}.

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Given a local observable X, we will generally specify a support for X, denoted by \mathcal{S}_X = [s_X, r_X]. \text{ We always assume } \emptyset \neq \mathcal{S}_X \subset [-L, L].
Consequences of droplet localization

Preliminaries

♦ $H = H_\omega$ will be a random XXZ spin chain satisfying droplet localization in the interval $I = I_{1,\delta} = \left[1 - \frac{1}{\Delta}, (2 - \delta)(1 - \frac{1}{\Delta})\right]$.

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Dynamical localization in the disordered XXZ spin chain
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♦ Given two local observables $X, Y$ we set $\text{dist}(X,Y) = \text{dist}(S_X, S_Y)$.
The time evolution of a local observable $X$ under $H^{(L)}$ is given by

$$\tau_t(X) = \tau_t^{(L)}(X) = e^{itH^{(L)}} X e^{-itH^{(L)}} \text{ for } t \in \mathbb{R}.$$
Time evolution in an energy window

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Thus, given an energy interval $J$, we consider the sub-Hilbert space Ran $P_j^{(L)}$, spanned by the the eigenstates of $H^{(L)}$ with energies in $J$, and localize an observable $X$ in the energy interval $J$ by considering its restriction to Ran $P_j^{(L)}$, 

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The time evolution of a local observable $X$ under $H^{(L)}$ is given by

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Since we only have localization in the energy interval $I$, and hence also in $I_0$, we should only expect manifestations of dynamical localization in these energy intervals.

Thus, given an energy interval $J$, we consider the sub-Hilbert space $\text{Ran} \ P^{(L)}_J$, spanned by the eigenstates of $H^{(L)}$ with energies in $J$, and localize an observable $X$ in the energy interval $J$ by considering its restriction to $\text{Ran} \ P^{(L)}_J$,

$$X_J = P^{(L)}_J XP^{(L)}_J.$$

Clearly

$$\tau_t(X_J) = (\tau_t(X))_J.$$
Non-spreading of information in the interval $I_0$
Theorem

There exists $C < \infty$, independent of $L$, such that for all local observables $X$, $t \in \mathbb{R}$ and $\ell > 0$, 

$$E(\sup_{t \in \mathbb{R}} \|X(\ell)(t) - \tau_t X\|_1) \leq C \|X\| e^{-\frac{1}{16} m \ell}.$$ 

$X_\ell = (X_{\ell})_\ell \Rightarrow$ the theorem holds with $I$ substituted for $I_0$. 

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Dynamical localization in the disordered XXZ spin chain
Theorem

There exists $C < \infty$, independent of $L$, such that for all local observables $X$, $t \in \mathbb{R}$ and $\ell > 0$, there is a local observable $X_\ell(t) = (X_\ell(t))_\omega$ with support $S_{X,\ell}$.
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$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\Vert (X_\ell(t) - \tau_t(X))_{I_0} \right\Vert_1 \right) \leq C \Vert X \Vert e^{-\frac{1}{16} m \ell}.$$
Non-spreading of information in the interval $I_0$

**Theorem**

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$$

$X_I = (X_{I_0})_I \implies$ the theorem holds with $I$ substituted for $I_0$. 
Zero-velocity Lieb-Robinson bounds

Theorem

The following holds uniformly in $L$:

$$E \left( \sup_{t \in \mathbb{R}} \| \tau_t (X_I, Y_I) \|_1 \right) \leq C \| X \| \| Y \| e^{-\frac{1}{8} m \text{dist} (X, Y)},$$

$$E \left( \sup_{t \in \mathbb{R}} \| \tau_t (X_{I_0}, Y_{I_0}) - \tau_t (X, P_0 Y - Y P_0) \|_1 \right) \leq C \| X \| \| Y \| e^{-\frac{1}{8} m \text{dist} (X, Y)},$$

$$E \left( \sup_{t, s \in \mathbb{R}} \| \left[ \tau_t (X_{I_0}), \tau_s (Y_{I_0}) \right], Z_{I_0} \|_1 \right) \leq C \| X \| \| Y \| \| Z \| e^{-\frac{1}{8} m \min \{ \text{dist} (X, Y), \text{dist} (X, Z), \text{dist} (Y, Z) \}}.$$

Moreover, the estimate (1) is not true without the counterterms.
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Zero-velocity Lieb-Robinson bounds

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$$
E \left( \sup_{t \in \mathbb{R}} \| [\tau_{t}(X_{l}), Y_{l}] \|_{1} \right) \leq C \| X \| \| Y \| e^{-\frac{1}{8}m \text{dist}(X, Y)},
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E \left( \sup_{t \in \mathbb{R}} \|[\tau_t (X_{I_0}), Y_{I_0}] - (\tau_t (X) P_0 Y - Y P_0 \tau_t (X))_I\|_1 \right) \leq C \|X\| \|Y\| e^{-\frac{1}{8} m \text{dist}(X,Y)},
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\[
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Correlators

We define the truncated time evolution of an observable $X$ in the energy window $I$ by ($H = H^{(L)}_\omega$),

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\tau^I_t (X) = e^{itH_I} X e^{-itH_I}, \quad \text{where} \quad H_I = H P_I.
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The correlator operator of two observables $X$ and $Y$ in the energy window $I$ is given by $(\bar{P}_I = 1 - P_I)$

$$R_I(X, Y) = P_I X \bar{P}_I Y P_I = (XY)_I - X_I Y_I.$$
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If $E$ is a simple eigenvalue with normalized eigenvector $\psi_E$, we have, with $R_E(X, Y) = R_{\{E\}}(X, Y)$,

$$\text{tr} \left( R_E(X, Y) \right) = \langle \psi_E, XY \psi_E \rangle - \langle \psi_E, X \psi_E \rangle \langle \psi_E, Y \psi_E \rangle.$$
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$$\text{tr}(R_E(X, Y)) = \langle \psi_E, XY \psi_E \rangle - \langle \psi_E, X \psi_E \rangle \langle \psi_E, Y \psi_E \rangle.$$

We are interested in quantities of the form $(K \subset I)$

$$R_K(\tau^I_t(X), Y) = \left(\tau^I_t(X) Y\right)_K - \left(\tau^I_t(X)\right)_K Y_K = \left(\tau^I_t(X) Y\right)_K - \tau_t(X_K) Y_K.$$
Dynamical exponential clustering

**Theorem**

*For all local observables $X$ and $Y$ we have, uniformly in $L$,***
Dynamical exponential clustering

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For all local observables $X$ and $Y$ we have, uniformly in $L$,

$$
\mathbb{E} \left( \sup_{t \in \mathbb{R}} \sum_{E \in \sigma_i(H(L))} | \text{tr} \left( R_E(\tau_t^l (X), Y) \right) | \right) \leq C \|X\| \|Y\| e^{-m \text{dist}(X,Y)},
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\[
E \left( \sup_{t \in \mathbb{R}} \sum_{E \in \sigma_i(H(L))} \left| \text{tr} \left( R_E (\tau_t (X_I), Y_I) \right) \right| \right) \leq C \| X \| \| Y \| e^{-m \text{dist}(X, Y)},
\]

and

\[
E \left( \sup_{t \in \mathbb{R}} \left| \text{tr} \left( R_{\tau_t^1} (X), Y) \right) \right| \right) \leq C \| X \| \| Y \| e^{-m \text{dist}(X, Y)}.
\]
General dynamical clustering

Theorem

Fix an interval $K = [1 - \Delta, \Theta] \subset I_1, \delta$, and $\alpha \in (0, 1)$. There exists $\tilde{m} > 0$, such that for all local observables $X$ and $Y$ we have, uniformly in $L$,

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} \| R_K(t \tau_K(X), Y) - (t \tau_K(X) P_0 Y + t \tau_K(Y) P_0 X) \| \right) \leq C \left( 1 + \ln \left( \min \{|S_X|, |S_Y|\} \right) \right) \|X\| \|Y\| e^{-\tilde{m} \text{dist}(X, Y)^\alpha}.$$

Moreover, the estimate is not true without the counterterms. While it is obvious where the first counterterm comes from, the same is not true of the second, where the time evolution seems to sit in the wrong place: it is $\tau_K(t \tau_K(Y))$ and not $\tau_K(t \tau_K(X))$. It turns out this term encodes information about the states above the energy window $K$, and the appearance of $\tau_K(t \tau_K(Y))$ is related to the reduction of this data to $P_0$. 

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$$\mathbb{E}\left(\sup_{t \in \mathbb{R}} \| R_K(\tau_K t(X)) Y - \tau_K t(X) P_0 Y + \tau_K t(Y) P_0 X \| \right) \leq C (1 + \ln(\min\{|S_X|, |S_Y|\})) \|X\| \|Y\| e^{-\tilde{m} \text{dist}(X, Y) \alpha}.$$ 

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Theorem

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$$\leq C \left( 1 + \ln \left( \min \{|S_X|, |S_Y|\} \right) \right) \|X\| \|Y\| e^{-\tilde{m} \text{dist}(X,Y)^\alpha}.$$

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\leq C \left( 1 + \ln \left( \min \{|S_X|, |S_Y|\} \right) \right) \|X\| \|Y\| e^{-\tilde{m} \text{dist}(X,Y)^\alpha}. 
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$$
\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\| R_{K} \left( \tau_{t}^{K} (X), Y \right) - \left( \tau_{t}^{K} (X) P_{0} Y + \tau_{t}^{K} (Y) P_{0} X \right)_{K} \right\| \right) \leq C \left( 1 + \ln \left( \min \{ |S_{X}|, |S_{Y}| \} \right) \right) \|X\| \|Y\| e^{-\tilde{m} \left( \text{dist}(X,Y) \right)^{\alpha}}.
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While it is obvious where the first counterterm comes from, the same is not true of the second, where the time evolution seems to sit in the wrong place: it is $\tau_{t}^{K} (Y)$ and not $\tau_{t}^{K} (X).$ It turns out this term encodes information about the states above the energy window $K$, and the appearance of $\tau_{t}^{K} (Y)$ is related to the reduction of this data to $P_{0}$. 
Decomposition of local observables

Given a local observable $X$, we define projections $P(X) = \bigotimes_{j \in S} X_1^2 (1 + \sigma_z^j)$ and $P(X) = 1 - P(S)$. Note that $P(X) \leq \sum_{i \in S} X_i N_i$ and $P(X) - P_0 = P_0 (P(X) - P_0) = 0$.

We have $X = \sum_{a, b \in \{+, -\}} X_{a, b}$, where $X_{a, b} = P(X) a P(X) b$. Moreover, since $P(X) +$ is a rank one projection on $H_S X$, we must have $X_+, X_+ = \zeta_X P(X)_+, \zeta_X \in \mathbb{C}$, $|\zeta_X| \leq \|X\|$. In particular, $(X - \zeta_X) +, + = 0$ and $\|X - \zeta_X\| \leq 2 \|X\|$, so we can assume $X_+, + = 0$ in the proofs.
Decomposition of local observables

Given a local observable $X$, we define projections $P^{(X)}_{\pm}$ by

$$P^{(X)}_+ = \bigotimes_{j \in S_x} \frac{1}{2}(1 + \sigma_j^z) \quad \text{and} \quad P^{(X)}_- = 1 - P^{(S)}_+.$$
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\]

Note that \( P^{(X)}_- \leq \sum_{i \in S_X} N_i \) and \( P^{(X)}_- P_0 = P_0 P^{(X)}_- = 0 \).
Decomposition of local observables

Given a local observable $X$, we define projections $P_{\pm}^{(X)}$ by

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Note that $P_-^{(X)} \leq \sum_{i \in S_X} N_i$ and $P_-^{(X)} P_0 = P_0 P_-^{(X)} = 0$.

We have $X = \sum_{a,b \in \{+, -\}} X^{a,b}$, where $X^{a,b} = P_a^{(X)} X P_b^{(X)}$.
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We have $X = \sum_{a,b \in \{+,-\}} X^{a,b}$, where $X^{a,b} = P_{a}^{(X)} X P_{b}^{(X)}$.

Moreover, since $P_{+}^{(X)}$ is a rank one projection on $\mathcal{H}_{S_X}$, we must have

$$X^{+,+} = \zeta_X P_{+}^{(X)}, \quad \text{where} \quad \zeta_X \in \mathbb{C}, \ |\zeta_X| \leq \|X\|.$$
Some flavor of the proofs

Decomposition of local observables

Given a local observable $X$, we define projections $P_{\pm}^{(X)}$ by

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In particular,

$$(X - \zeta_X)^{+,+} = 0 \quad \text{and} \quad \|X - \zeta_X\| \leq 2 \|X\|,$$

so we can assume $X^{+,+} = 0$ in the proofs.
**Consequences of droplet localization**

**Lemma**

Let $X, Y$ be local observables, $\ell \geq 1$. Then

\[
E \left( \sup_{g \in G_{I_0}} \left\| P_-(X) g(H) P_-(Y) \right\|_1 \right) \leq C e^{-m \text{dist}(X,Y)}
\]

\[
E \left( \left\| P_-(Y) P_-(X) P_{I_0} \right\|_1 \right) \leq C e^{-\frac{1}{2} m \text{dist}(X,Y)}
\]

\[
E \left( \sup_{I \in G_I} \left\| P_-(X) g(H) P_+(S_{X,I,\ell}) \right\|_1 \right) \leq C e^{-m \ell}
\]

\[
E \left( \sup_{g \in G_I} \left\| P_+(S_{Y,\ell}^c) g(H) P_+^{(S_{X,\ell}^c)} \right\|_1 \right) \leq C e^{-m(\text{dist}(X,Y) - 2\ell)}
\]
Non-spreading of information- Sketch of proof

To prove: Given a local observables \( X, t \in \mathbb{R} \) and \( \ell > 0 \), there is a local observable \( X_\ell(t) = (X_\ell(t))_\omega \) with support \( S_{X,\ell} \) satisfying

\[
\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\| (X_\ell(t) - \tau_t(X))_l \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} m \ell}.
\]
Non-spreading of information - Sketch of proof

To prove: Given a local observables $X$, $t \in \mathbb{R}$ and $\ell > 0$, there is a local observable $X_\ell(t) = (X_\ell(t))_\omega$ with support $S_{X,\ell}$ satisfying

$$
\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\| (X_\ell(t) - \tau_t(X))_{I_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16}m\ell}.
$$

Sketch of proof: Let $S_X = [s_X, r_X]$, recall $S_{X,\ell} = [s_X - \ell, r_X + \ell]$, and set

$$
\mathcal{O} = [-L, L] \setminus S_{X,\ell} = [-L, s_X - \ell) \cup (r_X + \ell, L]
$$

$$
\mathcal{T} = S_{X,\ell} \cap \mathcal{O} = [s_X - \ell, s_X - \ell) \cup (r_X + \ell, r_X + \ell]
$$
Some flavor of the proofs

Non-spreading of information- Sketch of proof

To prove: Given a local observables $X$, $t \in \mathbb{R}$ and $\ell > 0$, there is a local observable $X_\ell(t) = (X_\ell(t))_\omega$ with support $S_{X,\ell}$ satisfying

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$$

Sketch of proof: Let $S_X = [s_X, r_X]$, recall $S_{X,\ell} = [s_X - \ell, r_X + \ell]$, and set

$$
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$$

$$
\mathcal{T} = S_{X,\ell} \cap \mathcal{O} = [s_X - \ell, s_X - \frac{\ell}{2}) \cup (r_X + \frac{\ell}{2}, r_X + \ell]
$$

We first prove that

$$
\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\| \left( P_+^{(\mathcal{O})} \tau_t (X_{I_0}) P_+^{(\mathcal{O})} - \tau_t (X) \right)_{I_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} \ell \text{m}}.
$$
We now observe that for all observables $Z$ we have

$$P^{(\mathcal{O})}_+ Z P^{(\mathcal{O})}_+ = \tilde{Z} P^{(\mathcal{O})}_+ = P^{(\mathcal{O})}_+ \tilde{Z},$$

where $\tilde{Z}$ is an observable with $S_{\tilde{Z}} = S_X, \frac{\ell}{2}$ and $\|\tilde{Z}\| \leq \|Z\|$. 
We now observe that for all observables $Z$ we have

$$P_+^{(O)} Z P_+^{(O)} = \tilde{Z} P_+^{(O)} = P_+^{(O)} \tilde{Z},$$

where $\tilde{Z}$ is an observable with $S_{\tilde{Z}} = S_{X, \frac{\ell}{2}}$ and $\|\tilde{Z}\| \leq \|Z\|$. We conclude that

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\| \left( P_+^{(O)} \tau_t (X_{I_0}) - \tau_t (X) \right)_{I_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} m \ell}.$$
Some flavor of the proofs

We now observe that for all observables $Z$ we have

$$P_{+}^{(\mathcal{O})} Z P_{+}^{(\mathcal{O})} = \tilde{Z} P_{+}^{(\mathcal{O})} = P_{+}^{(\mathcal{O})} \tilde{Z},$$

where $\tilde{Z}$ is an observable with $S_{\tilde{Z}} = S_{X, \ell}^{2}$ and $\|\tilde{Z}\| \leq \|Z\|$. We conclude that

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\| \left( P_{+}^{(\mathcal{O})} \tau_{t} (X_{l_{0}}) - \tau_{t} (X) \right)_{l_{0}} \right\|_{1} \right) \leq C \|X\| e^{-\frac{1}{16} m \ell}.$$

Since $P_{+}^{(\mathcal{O})} \tau_{t} (X_{l_{0}})$ does not have support in $S_{X, \ell}$, we now define

$$X_{\ell}(t) = P_{+}^{(T)} \tau_{t} (X_{l_{0}}) \quad \text{for} \quad t \in \mathbb{R},$$

an observable with support in $S_{X, \ell}^{2} \cup T = S_{X, \ell}$.
We now observe that for all observables $Z$ we have

$$P_+^{(O)} Z P_+^{(O)} = \tilde{Z} P_+^{(O)} = P_+^{(O)} \tilde{Z},$$

where $\tilde{Z}$ is an observable with $S_{\tilde{Z}} = S_{X, \ell/2}$ and $\|\tilde{Z}\| \leq \|Z\|$.

We conclude that

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\| \left( P_+^{(O)} \tau_t (X_{I_0}) - \tau_t (X) \right)_{l_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{16} m \ell}.$$

Since $P_+^{(O)} \tau_t (X_{I_0})$ does not have support in $S_{X, \ell}$, we now define

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an observable with support in $S_{X, \ell/2} \cup \mathcal{T} = S_{X, \ell}$, and prove

$$\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left\| \left( P_+^{(O)} \tau_t (X_{I_0}) - X_{\ell}(t) \right)_{l_0} \right\|_1 \right) \leq C \|X\| e^{-\frac{1}{4} m \ell}.$$
The following lemma is an adaptation of an argument of Hastings, which combines the Lieb-Robinson bound with estimates on Fourier transforms.

**Lemma**

Let \( \alpha \in (0, 1) \), and consider a function \( f \in C^\infty_c(\mathbb{R}) \) such that

\[
|\hat{f}(t)| \leq C f e^{-m |t|^{\alpha}}
\]

for all \(|t| \geq 1\).

Then for all local observables \( X \) and \( Y \) we have, uniformly in \( L \),

\[
\|X f(H) Y - \int e^{-irH} Y \tau_r(X) \hat{f}(r) \, dr\| \leq C_1 \|X\| \|Y\| \left(1 + \|\hat{f}\|_1 e^{-m (\text{dist}(X, Y))^\alpha}\right).
\]

The commutator can be estimated by the Lieb-Robinson bound.
The following lemma is an adaptation of an argument of Hastings, which combines the Lieb-Robinson bound with estimates on Fourier transforms.

**Lemma**

Let $\alpha \in (0, 1)$, and consider a function $f \in C^\infty_c(\mathbb{R})$ such that

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The following lemma is an adaptation of an argument of Hastings, which combines the Lieb-Robinson bound with estimates on Fourier transforms.

**Lemma**

Let \( \alpha \in (0, 1) \), and consider a function \( f \in C_c^\infty(\mathbb{R}) \) such that

\[
|\hat{f}(t)| \leq C_f e^{-m_f |t|^\alpha} \quad \text{for all} \quad |t| \geq 1.
\]

Then for all local observables \( X \) and \( Y \) we have, uniformly in \( L \),

\[
\langle X f(H) Y - \int_{\mathbb{R}} e^{-i r H} Y \tau_r(X) \hat{f}(r) \rangle \leq C_1 \|X\| \|Y\| \left( 1 + \|\hat{f}\|_1 \right) e^{-m_1 \mathrm{dist}(X, Y)^\alpha}.
\]

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The following lemma is an adaptation of an argument of Hastings, which combines the Lieb-Robinson bound with estimates on Fourier transforms.

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Let $\alpha \in (0, 1)$, and consider a function $f \in C_c^\infty (\mathbb{R})$ such that

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Then for all local observables $X$ and $Y$ we have, uniformly in $L$,

$$\left\| Xf(H)Y - \int_{\mathbb{R}} e^{-irH} Y \tau_r (X) \hat{f}(r) dr \right\| \leq C_1 \|X\| \|Y\| \left( 1 + \left\| \hat{f} \right\|_1 \right) e^{-m_1 (\text{dist}(X,Y))\alpha}.$$
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\[
\left\| Xf(H)Y - \int_{\mathbb{R}} e^{-irH} Y \tau_r(X) \hat{f}(r) \, dr \right\|
\leq C_1 \|X\| \|Y\| \left(1 + \left\|\hat{f}\right\|_1\right) e^{-m_1\text{dist}(X,Y)\alpha}.
\]

\[
Xf(H)Y - \int_{\mathbb{R}} e^{-irH} Y \tau_r(X) \hat{f}(r) \, dr = \int_{\mathbb{R}} e^{-irH} [\tau_r(X), Y] \hat{f}(r) \, dr.
\]
The following lemma is an adaptation of an argument of Hastings, which combines the Lieb-Robinson bound with estimates on Fourier transforms.

**Lemma**

Let $\alpha \in (0, 1)$, and consider a function $f \in C_c^\infty(\mathbb{R})$ such that

$$\left| \hat{f}(t) \right| \leq C_f e^{-m_f |t|^\alpha} \text{ for all } |t| \geq 1.$$ 

Then for all local observables $X$ and $Y$ we have, uniformly in $L$,

$$\left\| Xf(H)Y - \int_{\mathbb{R}} e^{-irH} Y \tau_r(X) \hat{f}(r) \, dr \right\| \leq C_1 \|X\| \|Y\| \left( 1 + \left\| \hat{f} \right\|_1 \right) e^{-m_1 (\text{dist}(X,Y))^{\alpha}}.$$ 

$$Xf(H)Y - \int_{\mathbb{R}} e^{-irH} Y \tau_r(X) \hat{f}(r) \, dr = \int_{\mathbb{R}} e^{-irH} [\tau_r(X), Y] \hat{f}(r) \, dr.$$ 

The commutator can be estimated by the Lieb-Robinson bound.
Lemma

Let $K = [\Theta_0, \Theta_2]$ and $f \in C_c^\infty(\mathbb{R})$ with $\text{supp } f \subset [a_f, b_f]$. 
Lemma

Let $K = [\Theta_0, \Theta_2]$ and $f \in C_c^\infty(\mathbb{R})$ with $\text{supp} f \subset [a_f, b_f]$. Then for all local observables $X$ and $Y$ we have

$$\int_{\mathbb{R}} \left( e^{-irH} Y \tau_r (X) \right)_K \hat{f}(r) \, dr = \int_{\mathbb{R}} \left( e^{-irH} Y \{ P_K f \} \tau_r (X) \right)_K \hat{f}(r) \, dr,$$
**Lemma**

Let $K = [\Theta_0, \Theta_2]$ and $f \in C^\infty_c(\mathbb{R})$ with $\text{supp } f \subset [a_f, b_f]$. Then for all local observables $X$ and $Y$ we have

$$
\int_{\mathbb{R}} \left( e^{-irH} Y \tau_r(X) \right)_K \hat{f}(r) \, dr = \int_{\mathbb{R}} \left( e^{-irH} Y \{ P_{K_f} \} \tau_r(X) \right)_K \hat{f}(r) \, dr,
$$

where

$$
K_f = K + K - \text{supp } f \subset [2\Theta_0 - b_f, 2\Theta_2 - a_f].
$$
**Lemma**

Let \( K = [\Theta_0, \Theta_2] \) and \( f \in C^\infty_c(\mathbb{R}) \) with \( \text{supp} \, f \subset [a_f, b_f] \). Then for all local observables \( X \) and \( Y \) we have

\[
\int_{\mathbb{R}} \left( e^{-irH} Y \tau_r (X) \right)_K \hat{f}(r) \, dr = \int_{\mathbb{R}} \left( e^{-irH} Y \{ P_{K_f} \} \tau_r (X) \right)_K \hat{f}(r) \, dr,
\]

where

\[
K_f = K + K - \text{supp} \, f \subset [2\Theta_0 - b_f, 2\Theta_2 - a_f].
\]

For \( E, E' \in K \) we have

\[
P_E \left( \int_{\mathbb{R}} e^{-irH} Y \tau_r (X) \hat{f}(r) \, dr \right) P_{E'} = P_E Y f(E + E' - H) X P_{E'}
\]

\[
= P_E Y P_{K_f} f(E + E' - H) X P_{E'} = P_E \left( \int_{\mathbb{R}} e^{-irH} Y \{ P_{K_f} \} \tau_r (X) \hat{f}(r) \, dr \right) P_{E'}.
\]
Interval for droplet localization - Sketch of proof

To prove: Droplet localization in $I = [1 - \frac{1}{\Delta}, \Theta_1] \implies \Theta_1 \leq 2(1 - \frac{1}{\Delta})$. 

Sketch of proof: Let $\Theta_0 = 1 - \frac{1}{\Delta}$ and suppose $\Theta_1 > 2\Theta_0$. Let $K = [\Theta_0, \Theta_2]$, where $\Theta_0 < \Theta_2 < \Theta_1$, and $\epsilon = \min\{\Theta_1 - 2\Theta_2, \Theta_0\} > 0$. Fix a Gevrey class function $h$ such that $0 \leq h \leq 1$, $\text{supp} h \subset (-\epsilon, \epsilon)$, $h(0) = 1$, and $\|\hat{h}(t)\| \leq C e^{-c|t|^{1/2}}$. Note that $P_0 = h(H)$. Let $X, Y$ be local observables with $X + X^\dagger = Y + Y^\dagger = 0$. The Lemmas yield $\|XP_0Y\|_K \leq C\|X\|\|Y\| e^{-m^1(\text{dist}(X, Y))^{1/2}} + C' \sup_{r \in \mathbb{R}} \|YP_{h\tau}h^\tau(X)\|_K$, where $K_h \subset [2\Theta_0 - \epsilon, 2\Theta_2 + \epsilon] \subset [\Theta_0, \Theta_1] = I$. 

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Interval for droplet localization- Sketch of proof

To prove: Droplet localization in \( I = \left[ 1 - \frac{1}{\Delta}, \Theta_1 \right] \implies \Theta_1 \leq 2 \left( 1 - \frac{1}{\Delta} \right) \).

Sketch of proof: Let \( \Theta_0 = 1 - \frac{1}{\Delta} \) and suppose \( \Theta_1 > 2 \Theta_0 \). Let \( K = [\Theta_0, \Theta_2] \), where \( \Theta_0 < \Theta_2 < \Theta_1 \), and \( \varepsilon = \min \{ \Theta_1 - 2 \Theta_2, \Theta_0 \} > 0 \).
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Fix a Gevrey class function $h$ such that

$$0 \leq h \leq 1, \text{ supp } h \subset (-\varepsilon, \varepsilon), \ h(0) = 1, \text{ and } \left| \hat{h}(t) \right| \leq C e^{-c|t|^{\frac{1}{2}}}$$
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Sketch of proof: Let $\Theta_0 = 1 - \frac{1}{\Delta}$ and suppose $\Theta_1 > 2\Theta_0$. Let $K = [\Theta_0, \Theta_2]$, where $\Theta_0 < \Theta_2 < \Theta_1$, and $\varepsilon = \min \{ \Theta_1 - 2\Theta_2, \Theta_0 \} > 0$. Fix a Gevrey class function $h$ such that

$$0 \leq h \leq 1, \text{ supp } h \subset (-\varepsilon, \varepsilon), \quad h(0) = 1, \quad \text{and } \left| \hat{h}(t) \right| \leq Ce^{-c|t|^{\frac{1}{2}}}$$

Note that $P_0 = h(H)$. 
Interval for droplet localization- Sketch of proof

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Fix a Gevrey class function $h$ such that

$$0 \leq h \leq 1, \supp h \subset (-\varepsilon, \varepsilon), \ h(0) = 1, \ \text{and} \ \left| \hat{h}(t) \right| \leq Ce^{-c|t|^\frac{1}{2}}$$

Note that $P_0 = h(H)$.

Let $X, Y$ be local observables with $X^{+,+} = Y^{+,+} = 0$. The Lemmas yield

$$\| (XP_0 Y)_K \| = \| (Xh(H)Y)_K \| \leq C \| X \| \| Y \| e^{-m_1(\text{dist}(X,Y))^{\frac{1}{2}}} + C' \sup_{r \in \mathbb{R}} \| (YP_{K_h r} (X))_K \|, \quad$$

where $K_h \subset [2\Theta_0 - \varepsilon, 2\Theta_2 + \varepsilon] \subset [\Theta_0, \Theta_1] = I$. 

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Some flavor of the proofs

We can prove

\[ \mathbb{E} \left( \sup_{r \in \mathbb{R}} \left\| (YP_{K, r}(X))_K \right\| \right) \leq C \left\| X \right\| \left\| Y \right\| e^{-\frac{1}{8}m \text{dist}(X,Y)}, \]
We can prove

$$
E \left( \sup_{r \in \mathbb{R}} \| (Y P_{K_h T_r (X)})_K \| \right) \leq C \| X \| \| Y \| e^{-\frac{1}{8} m \text{dist}(X, Y)},
$$

so we conclude that

$$
E \left( \| (XP_0 Y)_K \| \right) \leq C \| X \| \| Y \| e^{-m_2 \text{dist}(X, Y)} \frac{1}{2}.
$$
We can prove

$$\mathbb{E} \left( \sup_{r \in \mathbb{R}} \| (YP_{K_h} \tau_r (X))_K \| \right) \leq C \| X \| \| Y \| e^{-\frac{1}{8} m \text{dist}(X,Y)},$$

so we conclude that

$$\mathbb{E} (\| (XP_0 Y)_K \|) \leq C \| X \| \| Y \| e^{-m_2 \text{dist}(X,Y)^{\frac{1}{2}}}.$$ 

In particular, it follows that we have, uniformly in $L$,

$$\mathbb{E} \left( \| \left( \sigma^x_i P^{(L)}_0 \sigma^x_j \right)_K \| \right) \leq C e^{-m_2 |i-j|^{\frac{1}{2}}} \text{ for all } i, j \in [-L, L]. \quad (2)$$
We can prove

\[
\mathbb{E} \left( \sup_{r \in \mathbb{R}} \| (YP_{K_h} \tau_r (X))_K \| \right) \leq C \| X \| \| Y \| e^{-\frac{1}{8} m \text{dist}(X,Y)},
\]

so we conclude that

\[
\mathbb{E} \left( \| (XP_0 Y)_K \| \right) \leq C \| X \| \| Y \| e^{-m_2 (\text{dist}(X,Y))^{\frac{1}{2}}.
\]

In particular, it follows that we have, uniformly in \( L \),

\[
\mathbb{E} \left( \| (\sigma_i^x P_0^{(L)} \sigma_j^x)_K \| \right) \leq C e^{-m_2 (|i-j|)^{\frac{1}{2}}} \quad \text{for all} \quad i, j \in [-L, L]. \quad (2)
\]

But we can show that for all \( i, j \in \mathbb{Z} \) with \(|i - j| \geq R_K\), we have

\[
\mathbb{E} \left( \liminf_{L \to \infty} \| (\sigma_i^x P_0^{(L)} \sigma_j^x)_K \| \right) \geq \gamma_K > 0. \quad (3)
\]
We can prove

$$\mathbb{E} \left( \sup_{r \in \mathbb{R}} \left\| (YP_{K_h \tau_r (X)})_K \right\| \right) \leq C \left\| X \right\| \left\| Y \right\| e^{-\frac{1}{8} m \text{dist}(X, Y)} ,$$

so we conclude that

$$\mathbb{E} (\left\| (XP_0 Y)_K \right\|) \leq C \left\| X \right\| \left\| Y \right\| e^{-m_2(\text{dist}(X, Y))^{\frac{1}{2}}} .$$

In particular, it follows that we have, uniformly in $L$,

$$\mathbb{E} \left( \left\| \left( \sigma_i^x P_0^{(L)} \sigma_j^x \right)_K \right\| \right) \leq C e^{-m_2(|i-j|)^{\frac{1}{2}}} \text{ for all } i, j \in [-L, L]. \quad (2)$$

But we can show that for all $i, j \in \mathbb{Z}$ with $|i - j| \geq R_K$, we have

$$\mathbb{E} \left( \liminf_{L \to \infty} \left\| \left( \sigma_i^x P_0^{(L)} \sigma_j^x \right)_K \right\| \right) \geq \gamma_K > 0. \quad (3)$$

(2) and (3) give a contradiction $\Rightarrow \Theta_1 \leq 2\Theta_0$. 

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