The Classical XY-Model – *Vortex-gas- and Random Walk Representations*

Jürg Fröhlich, ETH Zurich

IHES, 10 février, 2017

Remerciments

In memory of the great times I lived at IHES – and with thanks to David Ruelle and Henri Epstein for all they contributed to an excellent quality of life I enjoyed during those years. – Thanks also to Hugo Duminil-Copin for inviting me to talk today.

Almost all the material presented in this lecture is the result of work I was involved in when I was a member of the IHES, between January 1978 and July 1982. [1] My main partner in this particular journey (and others) was my friend and mentor Tom Spencer (IAS). Interactions with David Brydges, Erhard Seiler and Barry Simon also played a useful role.

[1] During my $4\frac{1}{2}$ years at IHES, I mainly worked on Statistical Mechanics, QFT, and the Theory of Disordered Systems.

Contents

Summary

The three theorists who have won the *2016 Nobel Prize in Physics:*

David Thouless

Duncan Haldane

Mike Kosterlitz

A survey of phenomena special to Physics in 2D is presented.
The Mermin-Wagner- . . . theorem is recalled. Some crucial ideas – among others, *the vortex-gas representation, energy-entropy arguments for vortices, etc.* – in a proof of existence of the K-T transition in the 2D classical XY model are highlighted.

Subsequently, the R-W rep. of the XY model is introduced and used to study the behaviour of correlations in an external mag. field and to prove bounds on critical exponents for the magnetisation and the correlation length, as the external field tends to 0, etc.

1. Phase transitions and (absence of) symmetry breaking

To be specific, consider N-vector models: with each site x of $\mathbb{Z}^2$ associate a classical "spin", $\vec{S}_x \in S^{N-1}$, with Hamiltonian

$$H\left(\{\vec{S}_\cdot\}\right) := -J \sum_{\langle x,y \rangle} \vec{S}_x \cdot \vec{S}_y + h \sum_x S_x^1, \tag{1}$$

where J is the exchange coupling constant, and h is an external magnetic field in the 1-direction. The Gibbs state at inverse temperature β is defined by

$$\langle A\left(\{\vec{S}_\cdot\}\right)\rangle_{\beta,h} = Z_{\beta,h}^{-1} \int A\left(\{\vec{S}_\cdot\}\right) \cdot \exp[-\beta H(\{\vec{S}_\cdot\})] \times$$
$$\times \prod_{x \in \mathbb{Z}^2} \delta(|\vec{S}_x|^2 - 1)\, \mathrm{d}^N S_x, \tag{2}$$

where A is a local functional of $\{\vec{S}_x\}_{x \in \mathbb{Z}^2}$.

Phase transitions – or their absence

(i) For $N \leq 3$, Lee-Yang implies absence of phase transitions and exp. decay of conn. correlations whenever $h \neq 0$. What about $N > 3$?

(ii) $N = 1$, *Ising model*
Phase transition with order parameter driven by (im-)balance between energy and entropy of extended defects, the *Peierls contours*. Given $(S = -1)$-boundary cond.,

$$\text{Prob}\{S_0 = +1\} \leq \sum_{\substack{\text{contours } \gamma: \\ \text{int}\gamma \ni 0}} \exp[-\beta J |\gamma|],$$

$\Rightarrow$ spont. magnetization & spont.breaking of $S \to -S$ symmetry at low enough temps.! Interfaces between $+$ phase and $-$ phase always rough. (2D Ising model exactly solved by Onsager, Kaufman, Yang,. . . , Smirnov et al.,. . . ; RG fixed point: unitary CFT; SLE,...)

(iii) $N \geq 2$, classical XY- and Heisenberg models
Internal symmetry is $SO(N)$, (connected and continuous for $N \geq 2$). *Mermin-Wagner theorem* says that, in 2D, continuous symmetries of models with short-range ints. cannot be broken spontaneously.

Absence of symmetry breaking

Proof: Fisher's droplet picture made precise using *relative entropy*!

(iv) McBryan-Spencer bound: For $N = 2, h = 0$, use angular variables:

$$\vec{S}_j \cdot \vec{S}_k = \cos(\theta_j - \theta_k),\ H = -J \sum_{\langle j,k \rangle} \cos(\theta_j - \theta_k),\ \theta_j \in [0, 2\pi) \quad \Rightarrow$$

$$\langle \vec{S}_0 \cdot \vec{S}_x \rangle_\beta = Z_\beta^{-1} \int \prod_j \mathrm{d}\theta_j\ e^{i(\theta_0 - \theta_x)} \exp[\beta J \sum_{\langle j,k \rangle} \cos(\theta_j - \theta_k)]. \quad (3)$$

Let $C(j) \simeq -\frac{1}{2\pi} \ell n |j|$ (2D Coulomb potential) be the Green fct. of the discrete Laplacian: $-(\Delta C)(j) = \delta_{0j}$. Complex transl. in (3):

$$\theta_j \to \theta_j + i a_j, \quad \text{where} \quad a_j := (\beta J)^{-1} [C(j) - C(j - x)].$$

Using that

$$|\Re \cos(\theta_j - \theta_k + i(a_j - a_k)) - \cos(\theta_j - \theta_k)| \leq \frac{1}{2}(1 + \varepsilon)(a_j - a_k)^2,$$

for $\beta > \beta_0(\varepsilon)$, and $e^{-(a_0 - a_x)} < (|x| + 1)^{-(1/\pi\beta J)}$, we find:

Absence of symmetry breaking – ctd.

Theorem. (McBryan & Spencer)
In the XY Model ($N = 2$), given $\varepsilon > 0$, there is a $\beta_0(\varepsilon) < \infty$ such that, for $\beta \geq \beta_0(\varepsilon)$,

$$\langle \vec{S}_0 \cdot \vec{S}_x \rangle_{\beta, h=0} \leq (|x| + 1)^{-(1-\varepsilon)/(2\pi\beta J)} \tag{4}$$

i.e., conn. correlations are bounded above by inverse power laws. □

Remarks. (i) Using *Ginibre's correlation inequalities*, result extends to *all* N-vector models with $N \geq 2$. It also holds for quantum XY model, etc.
(ii) In Villain model, (4) holds for $\varepsilon = 0$, ($\nearrow$ vortex-gas rep.)!

Conjecture. (Polyakov) For $N \geq 3$, the 2D N-vector model is ultraviolet asymptotically free, and

$$\langle \vec{S}_0 \cdot \vec{S}_x \rangle_{\beta, h=0} \leq \text{const. } |x|^{-(1/2)} \exp\left[-m(\beta, N)|x|\right],$$

for some "mass gap" $m(\beta, N)$ which is positive $\forall \beta < \infty$. □
It has been proven (F-Spencer, using "infrared bounds") that

$$m(\beta, N) \leq \text{ const. } e^{-\mathcal{O}(\beta J/N)}.$$

2. Kosterlitz-Thouless transition

The following theorem proves a conjecture made by (Feynman, Berezinskii,...) *Kosterlitz* and *Thouless.*

Theorem. (F-Spencer; proof occupies $\geq$ 60 pages)
There exists a finite constant β_0 and a "dielectric constant" $0 < \epsilon(\beta) < 1$ such that, for $\beta > \beta_0$,

$$\langle \vec{S}_0 \cdot \vec{S}_x \rangle_{\beta, h=0} \geq \text{const. } (|x| + 1)^{-(1/2\pi\epsilon^2 \beta J)}, \tag{5}$$

with $\epsilon(\beta) \to 1$, as $\beta \to \infty$. $\square$

Remark. It is well known (and easy to prove) that if β is small enough $\langle \vec{S}_0 \cdot \vec{S}_x \rangle_{\beta, h=0}$ decays exp. fast in $|x|$. (This can be interpreted as *"Debye screening"* in a 2D Coulomb gas dual to the XY model.)

It is a little easier to analyze the *"Villain approx."* to the XY model. This model is "dual" (in the sense of Kramers & Wannier) to the so-called *"discrete Gaussian model"* used to study the *roughening transition* of 2D interfaces of integer height. Note that:

Kramers-Wannier duality $\underset{ess.}{\simeq}$ *Poincaré duality* for a 2D cell complex.

Kosterlitz-Thouless transition – ctd.

This (and $\pi_1(S^1) = \mathbb{Z}$) is extent to which *"topology"* plays a role in this story.

Using the *Poisson summation formula*, one shows that

$$\boxed{\textit{discrete Gaussian} \simeq \textit{2D Coulomb gas},}$$

with *charges* in Coulomb gas = *vortices* in XY- (or Villain) model.
For large T, the Coulomb gas is in a plasma phase of unbound charges ($\rightarrow$ Debye screening).

Multi-scale analysis (F-Spencer):

A purely combinatorial construction is used to rewrite the Coulomb gas (dual to Villain) as a convex combination of *gases of neutral multipoles* (dipoles, quadrupoles, etc.) of arbitrary diameter, with the property that a multipole ρ of diameter $d(\rho)$, (ρ being a charge distribution of total el. charge 0) is separated from other multipoles of *larger* diameter by a dist.

$$\geq \text{const.} d(\rho)^{\alpha}, \ \alpha \in \left(\tfrac{3}{2}, 2\right). \qquad (^*)$$

The "entropy" of a multipole ρ is denoted by $S(\rho)$. It is a purely combinatorial quantity indep. of β and is bd. above by $V(\rho)$, where $V(\rho)$ is a *"multi-scale volume"* of supp(ρ) adapted to $(^*)$..., (see sketch).

Kosterlitz-Thouless transition – ctd.

Now, using complex translations to derive rather intricate electrostatic inequalities that exploit $(^*)$, one shows that the self-energy, $E(\rho)$, of a neutral multipole with distribution ρ is bounded below by

$$\boxed{E(\rho) \geq c_1 \|\rho\|_2^2 + c_2 \ell n\, d(\rho) \geq c_3 V(\rho),} \tag{6}$$

where $c_i, i = 1, 2, 3$, are positive constants.
The bound (6) implies that the *"free energy"*,$F(\rho)$, of a neutral multipole with charge distribution ρ is bounded below by

$$F(\rho) > (1 - \varepsilon) E(\rho)$$

provided $\beta > \beta(\varepsilon)$, for some finite $\beta(\varepsilon)$. This implies that, for β large enough, neutral multipoles with charge distribution ρ of large (multi-scale) volume $V(\rho)$, and hence large electrostatic energy $E(\rho)$, have a *very tiny density;* (dipoles of small dipole moment dominate!).
The proof of the *Theorem* is completed by showing that dilute gases of neutral multipoles do not screen electric charges $\Rightarrow$ inverse power-law decay of spin-spin correlations, $\propto \exp[(\varepsilon^2 \beta J)^{-1} \times$ (Coulomb pot.)].

3. Remarks on peculiarities of Physics in 2D

1. A type of quantum statistics *not* anticipated by the founders of QM is braid (group) statistics, which only appears as statistics of fields in (1+1)-D QFT (1975), and as statistics of fields/particles in 2D systems, (1977, 1987). Particles in 2D with braid stat. always have fractional spin and often fract. electric charge. They are expected to exist as quasi-particles in 2DEG exhibiting the QHE. They may have applications to *topological quantum computation.*
2. Among quasi-particles in 2D quantum systems (graphene, topol. insulators) are ones that mimic, e.g., 2-component Dirac fermions, leading to phenomena such as an anomalous Hall effect.
3. General principles of quantum physics, such as gauge anomalies and their cancellations, bulk-edge duality, holography, etc. are manifested (with impact) in various 2D quantum many-body systems.
4. Huyghens' Principle – i.e., e.m. waves propagating along surface of light cones – is *violated* in 2D. This might imply that 2D systems are fundamentally *quantum*, without classical facets.

Next, I turn to vortex-gas and R-W representations of XY model (BB)!