Physics in 2D – *from the Kosterlitz-Thouless Transition to Topological Insulators*

Jürg Fröhlich, ETH Zurich

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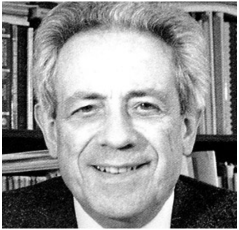
Dedicated to the memory of Louis Michel and Roland Sénéor – two 'Polytechniciens' who, at certain points, made interesting observations leading to some insights and results I will describe in the following. I remember them fondly.



1923 – 1999

1938–2016

Credits and Contents

I thank numerous companions on scientific journeys described in this lecture; in particular: *Thomas C. Spencer, Vaughan F. R. Jones, Thomas Kerler, Rudolf Morf, Urban Studer, Emmanuel Thiran, Gian Michele Graf, Johannes Walcher, Bill Pedrini, and Christoph Schweigert* – among others.

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Summary

Some parts of this lecture are related to work of three theorists who have won the *2016 Nobel Prize in Physics:*



David Thouless Duncan Haldane Mike Kosterlitz

A survey of "Physics in 2D" is presented:
The Mermin-Wagner- ... theorem is recalled. Crucial ideas – among others, *energy-entropy arguments for defects* – in a proof of existence of the K-T transition in the 2D classical XY model are highlighted.

Subsequently, the TFT-approach to the FQHE and to 2D time-reversal invariant topological insulators with chiral edge spin-currents (1993) is described. The roles of anomaly cancellation and of bulk-edge duality in the analysis of such systems are explained.

Part I. "Topological" Phase Transitions in 2D Systems

I.1 Phase transitions and (absence of) symmetry breaking.

To be specific, consider N-vector models: with each site x of $\mathbb{Z}^2$ associate a classical "spin", $\vec{S}_x \in S^{N-1}$, with Hamiltonian

$$H\left(\{\vec{S}_\cdot\}\right) := -J \sum_{\langle x,y\rangle} \vec{S}_x \cdot \vec{S}_y + h \sum_x S_x^1, \tag{1}$$

where J is the exchange coupling constant, and h is an external magnetic field in the 1-direction. The Gibbs state at inverse temperature β is defined by

$$\langle A\left(\{\vec{S}_\cdot\}\right)\rangle_{\beta,h} = Z_{\beta,h}^{-1} \int A\left(\{\vec{S}_\cdot\}\right) \cdot \exp[-\beta H(\{\vec{S}_\cdot\})] \times$$

$$\times \prod_{x\in\mathbb{Z}^2} \delta(|\vec{S}_x|^2 - 1)\, \mathrm{d}^N S_x, \tag{2}$$

where A is a local functional of $\{\vec{S}_x\}_{x\in\mathbb{Z}^2}$.

Phase transitions – or their absence

(i) In any dimension & for $N \leq 3$, Lee-Yang implies absence of phase transitions and exp. decay of conn. correlations whenever $h \neq 0$. What about $N > 3$?

(ii) $N = 1$, *Ising model*
Phase transition with order parameter at $h = 0$ driven by (im-)balance between energy and entropy of extended defects, the *Peierls contours*. Given $(S = -1)$-boundary cond.,

$$\mathrm{Prob}\{S_0 = +1\} \leq \sum_{\substack{\text{contours } \gamma: \\ \text{int}\gamma \ni 0}} \exp[-\beta J|\gamma|],$$

$\Rightarrow$ spont. magnetization & spont.breaking of $S \to -S$ symmetry at low enough temps.! Interfaces between $+$ phase and $-$ phase always rough. (2D Ising model exactly solved by Onsager, Kaufman, Yang,..., Smirnov et al.,...; RG fixed point: unitary CFT; SLE,...)

(iii) $N \geq 2$, classical XY- and Heisenberg models
At $h = 0$, internal symmetry is $SO(N)$, (connected & continuous for $N \geq 2$). *Mermin-Wagner theorem:* In 2D, continuous symmetries of models with short-range interactions *not* broken spontaneously.

Absence of symmetry breaking

Proof: Fisher's droplet picture made precise using *relative entropy*!

(iv) McBryan-Spencer bound: For $N = 2, h = 0$, use angular variables:

$$\vec{S}_j \cdot \vec{S}_k = \cos(\theta_j - \theta_k),\ H = -J \sum_{\langle j,k \rangle} \cos(\theta_j - \theta_k),\ \theta_j \in [0, 2\pi) \quad \Rightarrow$$

$$\langle \vec{S}_0 \cdot \vec{S}_x \rangle_\beta = Z_\beta^{-1} \int \prod_j \, \mathrm{d}\theta_j \ e^{i(\theta_0 - \theta_x)} \exp[\beta J \sum_{\langle j,k \rangle} \cos(\theta_j - \theta_k)]. \quad (3)$$

Let $C(j) \simeq -\frac{1}{2\pi} \ell n|j|$ (2D Coulomb potential) be the Green fct. of the discrete Laplacian: $-(\Delta C)(j) = \delta_{0j}$. Complex transl. in (3):

$$\theta_j \to \theta_j + i a_j, \quad \text{where} \quad a_j := (\beta J)^{-1}[C(j) - C(j - x)].$$

Using that

$$|\Re \cos(\theta_j - \theta_k + i(a_j - a_k)) - \cos(\theta_j - \theta_k)| \leq \frac{1}{2}(1 + \varepsilon)(a_j - a_k)^2,$$

for $\beta > \beta_0(\varepsilon)$, and $e^{-(a_0 - a_x)} < (|x| + 1)^{-(1/\pi\beta J)}$, we find:

Absence of symmetry breaking – ctd.

Theorem. (McBryan & Spencer, ...)
In the XY Model ($N = 2$), given $\varepsilon > 0$, there is a $\beta_0(\varepsilon) < \infty$ such that, for $\beta \geq \beta_0(\varepsilon)$,

$$\langle \vec{S}_0 \cdot \vec{S}_x \rangle_{\beta, h=0} \leq (|x| + 1)^{-(1-\varepsilon)/(2\pi\beta J)} \tag{4}$$

i.e., conn. correlations are bounded above by inverse power laws. $\square$

Remarks. (i) Using *Ginibre's correlation inequalities*, result extends to *all* N-vector models with $N \geq 2$. It also holds for quantum XY model, etc.
(ii) In Villain model, (4) holds for $\varepsilon = 0$, (by Kramers-Wannier duality)!

Conjecture. (Polyakov) For $N \geq 3$, the 2D N-vector model is ultraviolet asymptotically free, and

$$\langle \vec{S}_0 \cdot \vec{S}_x \rangle_{\beta, h=0} \leq \text{const. } |x|^{-(1/2)} \exp\left[-m(\beta, N)|x|\right],$$

for some "mass gap" $m(\beta, N)$ which is positive $\forall \beta < \infty$. $\square$
It has been proven (F-Spencer, using "infrared bounds") that

$$m(\beta, N) \leq \text{ const. } e^{-\mathcal{O}(\beta J/N)}.$$

Kosterlitz-Thouless transition

I.2 The Kosterlitz-Thouless transition in the 2D classical XY model

The following theorem proves a conjecture made by (Berezinskii,...) Kosterlitz and Thouless.

Theorem. (F-Spencer; proof occupies $\geq$ 60 pages)
There exists a finite constant β_0 and a "dielectric constant" $0 < \epsilon(\beta) < 1$ such that, for $\beta > \beta_0$,

$$\langle \vec{S}_0 \cdot \vec{S}_x \rangle_{\beta, h=0} \geq \text{const.}\ (|x| + 1)^{-(1/2\pi\epsilon^2\beta J)}, \tag{5}$$

with $\epsilon(\beta) \to 1$, as $\beta \to \infty$. $\square$

Remark. It is well known (and easy to prove) that if β is small enough $\langle \vec{S}_0 \cdot \vec{S}_x \rangle_{\beta, h=0}$ decays exp. fast in $|x|$. (This can be interpreted as *"Debye screening"* in a 2D Coulomb gas dual to the XY model.)

It is a little easier to analyze the *"Villain approx."* to the XY model. This model is "dual" (in the sense of Kramers & Wannier) to the so-called *"discrete Gaussian model"* used to study the *roughening transition* of 2D interfaces of integer height. Note that:

Kosterlitz-Thouless transition – ctd.

Kramers-Wannier duality $\underset{ess.}{\simeq}$ *Poincaré duality* for a 2D cell complex. This (and $\pi_1(S^1) = \mathbb{Z}$) is extent to which *"topology"* plays a role in this story.

Using the *Poisson summation formula*, one shows that

$$\boxed{\textit{discrete Gaussian} \simeq \textit{2D Coulomb gas},}$$

with *charges* in Coulomb gas = *vortices* in XY- (or Villain) model.
For large T, the Coulomb gas is in a plasma phase of unbound charges.

Multi-scale analysis (F-Spencer):

A purely combinatorial construction is used to rewrite the Coulomb gas (dual to Villain) as a convex combination of *gases of neutral multipoles* (dipoles, quadrupoles, etc.) of arbitrary diameter, with the property that a multipole ρ of diameter $d(\rho)$, (ρ being a charge distribution of total el. charge 0) is separated from other multipoles of *larger* diameter by a dist.

$$\geq \text{const.} d(\rho)^{\alpha}, \;\; \alpha \in \left(\tfrac{3}{2}, 2\right). \qquad (^*)$$

The "entropy" of a multipole ρ is denoted by $S(\rho)$. It is a purely combinatorial quantity indep. of β and is bd. above by $V(\rho)$, where $V(\rho)$ is a *"multi-scale volume"* of supp(ρ) adapted to $(^*)$.

Kosterlitz-Thouless transition – ctd.

Now, using complex translations to derive rather intricate electrostatic inequalities that exploit (*), one shows that the self-energy, $E(\rho)$, of a neutral multipole with distribution ρ is bounded below by

$$\boxed{E(\rho) \geq c_1 \|\rho\|_2^2 + c_2 \ell n \, d(\rho) \geq c_3 V(\rho),} \tag{6}$$

where $c_i, i = 1, 2, 3$, are positive constants.
The bound (6) implies that the *"free energy"*, $F(\rho)$, of a neutral multipole with charge distribution ρ is bounded below by

$$F(\rho) > (1 - \varepsilon) E(\rho)$$

provided $\beta > \beta(\varepsilon)$, for some finite $\beta(\varepsilon)$. This implies that, for β large enough, neutral multipoles with charge distribution ρ of large (multi-scale) volume $V(\rho)$, and hence large electrostatic energy $E(\rho)$, have a *very tiny density;* (dipoles of small dipole moment dominate!).
The proof of the *Theorem* is completed by showing that dilute gases of neutral multipoles do not screen electric charges $\Rightarrow$ inverse power-law decay of spin-spin correlations, $\propto \exp[(\varepsilon^2 \beta J)^{-1} \times (\text{Coulomb pot.})]$.

Braid statistics, violation of Huyghens' Principle, etc.

I.3 Survey of phenomena special to Physics in 2D

1. A type of quantum statistics *not* anticipated by the founders of QM is braid (group) statistics, which only appears as statistics of fields in (1+1)-D QFT (1975), and as statistics of fields/particles in 2D systems, (1977, 1987). Particles in 2D with braid stat. always have fractional spin and often fract. electric charge. They are expected to exist as quasi-particles in 2DEG exhibiting the QHE – see Part II. They may have applications to *topological quantum computation.*

2. Among quasi-particles in 2D quantum systems (graphene, topol. insulators) are ones that mimic, e.g., 2-component Dirac fermions, leading to phenomena such as an anomalous Hall effect; $\nearrow$ Part II.

3. General principles of quantum physics, such as gauge anomalies and their cancellations, bulk-edge duality, holography, etc. are manifested (with impact) in various 2D quantum many-body systems.

4. Huyghens' Principle – i.e., e.m. waves propagating along surface of light cones – is *violated* in 2D. This might imply that 2D systems are fundamentally *quantum*, without classical facets.

Part II. What Topological Field Theory Tells Us About the FQHE and Topological Insulators

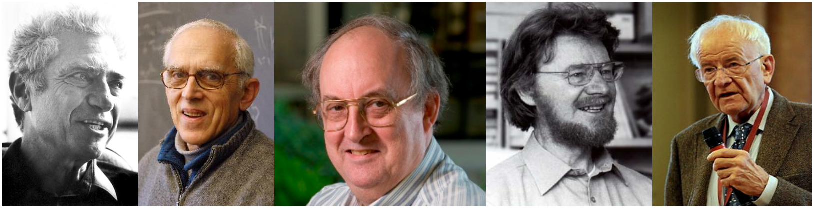
General goals of analysis

- Classify bulk- and surface states of (condensed) matter, using concepts and results from gauge theory, current algebra & GR: Effective actions (= generating functionals of connected current Green fcts.! transport coefficients!), gauge-invariance, anomalies & their cancellation, "holography", etc.
- Extend Landau Theory of Phases and Phase Transitions to a Gauge Theory of Phases of Matter.

Applications

- Fractional Quantum Hall Effect (1989 – 2012)
- Topological Insulators and -Superconductors (1994 – 2015)
- Higher-dimensional cousins of QHE $\Rightarrow$ Cosmology: Primordial magnetic fields in the Universe, matter-antimatter asymmetry, dark matter & dark energy, etc. (2000 – $\cdots$)

The chiral anomaly



Anomalous axial currents (for massless fermions):
In 2D:

$$\partial_\mu j_5^\mu = \frac{\alpha}{2\pi} E, \quad \alpha := \frac{e^2}{\hbar}, \quad [j_5^0(\vec{y}, t), j^0(\vec{x}, t)] \overset{(ACC)}{=} i\alpha\delta'(\vec{x} - \vec{y})$$

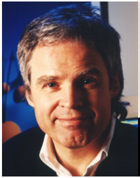
In 4D:

$$\partial_\mu j_5^\mu = \frac{\alpha}{\pi} \vec{E} \cdot \vec{B},$$

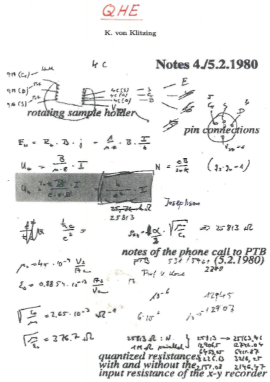
and

$$[j_5^0(\vec{y}, t), j^0(\vec{x}, t)] \overset{(ACC)}{=} i\frac{\alpha}{\pi} \vec{B}(\vec{y}, t) \cdot \nabla_{\vec{y}} \delta(\vec{x} - \vec{y})$$

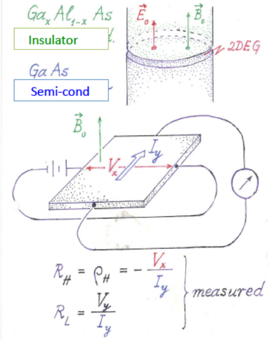
1. Anomalous Chiral Edge Currents in Incomp. Hall Fluids

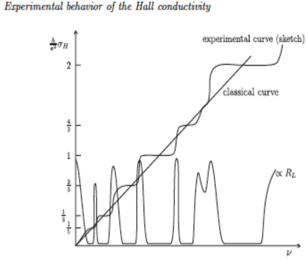



From von Klitzing's lab journal ($\Rightarrow$ 1985 Nobel Prize in Physics):



Setup & basic quantities





2D EG confined to $\Omega \subset xy$ - plane , in mag. field $\vec{B}_0 \perp \Omega$; ν such that $R_L = 0$. Response of 2D EG to small perturb. em field, $\vec{E} \| \Omega$, $\vec{B} \perp \Omega$, with $\vec{B}^{tot} = \vec{B}_0 + \vec{B}$, $\quad B := |\vec{B}|$, $\quad \underline{E} := (E_1, E_2)$.

Field tensor: $\quad F := \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & -B \\ -E_2 & B & 0 \end{pmatrix} = dA, \quad$ (A: vector pot.)

Electrodynamics of 2D incompressible e^--gases

Def.:

$$j^\mu(x) = \langle J^\mu(x)\rangle_A, \quad \mu = 0, 1, 2.$$

(1) Hall's Law

$$\underline{j}(x) = \sigma_H \big(\underline{E}(x)\big)^*, \quad (R_L = 0!) \;\to\; \text{broken } P,\, T \tag{1}$$

(2) Charge conservation

$$\frac{\partial}{\partial t}\rho(x) + \underline{\nabla} \cdot \underline{j}(x) = 0 \tag{2}$$

(3) Faraday's induction law

$$\frac{\partial}{\partial t} B_3^{tot} + \underline{\nabla} \wedge \underline{E}(x) = 0 \tag{3}$$

Then

$$\frac{\partial \rho}{\partial t} \overset{(2)}{=} -\underline{\nabla} \cdot \underline{j} \overset{(1)}{=} -\sigma_H \underline{\nabla} \wedge \underline{E} \overset{(3)}{=} \sigma_H \frac{\partial B}{\partial t} \tag{4}$$

ED of 2D e^--gases, ctd.

Integrate (4) in t, with integration constants chosen as follows:

$$j^0(x) := \rho(x) + e \cdot n, \;\; B(x) = B_3^{tot}(x) - B_0 \qquad \Rightarrow$$

(4) Chern-Simons Gauss law

$$j^0(x) = \sigma_H B(x) \tag{5}$$

$$\text{Eqs. (1) and (5)} \quad \Rightarrow \boxed{j^\mu(x) = \sigma_H \; \varepsilon^{\mu\nu\lambda} \; F_{\nu\lambda}(x)} \tag{6}$$

Now

$$0 \overset{(2)}{=} \partial_\mu j^\mu \overset{(3),(6)}{=} \varepsilon^{\mu\nu\lambda}(\partial_\mu \sigma_H) F_{\nu\lambda} \;\; \neq 0, \tag{7}$$

wherever $\sigma_H \neq$ *const*., e.g., at $\partial\Omega$. – Actually, j^μ is *bulk* current density, (j^μ_{bulk}), $\neq$ conserved *total* electric current density:

$$j^\mu_{tot} = j^\mu_{bulk} + j^\mu_{edge}, \quad \partial_\mu j^\mu_{tot} = 0, \;\text{but}\;\; \partial_\mu j^\mu_{bulk} \overset{(7)}{\neq} 0 \tag{8}$$

Anomalous chiral edge currents

We have that

$$\text{supp } j^\mu_{edge} = \text{supp}(\underline{\nabla}\sigma_H) \supseteq \partial\Omega, \qquad \underline{j}_{edge} \perp \underline{\nabla}\sigma_H.$$

"Holography": On $\text{supp}(\underline{\nabla}\sigma_H)$,

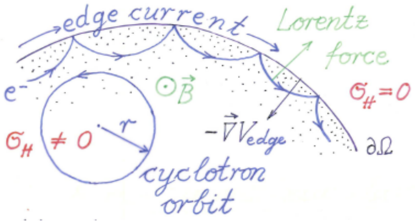
$$\boxed{\partial_\mu j^\mu_{edge} \overset{(8)}{=} -\partial_\mu j^\mu_{bulk}|_{\text{supp}(\underline{\nabla}\sigma_H)} \overset{(6)}{=} -\sigma_H E_\parallel|_{\text{supp}(\underline{\nabla}\sigma_H)}} \qquad (9)$$

Chiral anomaly in 1+1 dimensions!

Edge current, $j^\mu_{edge} \equiv j^\mu_5$, is anomalous chiral current in $1+1$ D: At edge,

$$\frac{e}{c} B^{tot} v_\parallel = (\underline{\nabla} V_{edge})^*, \quad V_{edge}: \text{ confining edge pot.}$$

Skipping orbits, hurricanes and fractional charges



Analogous phenomenon in classical physics: *Hurricanes!*

$\vec{B} \to \vec{\omega}_{earth}$, Lorentz force $\to$ Coriolis force , $\underline{\nabla} V_{edge} \to \underline{\nabla}$ pressure .

Chiral anomaly in $(1+1)$D:

$$\partial_\mu j_5^\mu = -\frac{e^2}{h}\Big(\sum_{species\ \alpha} Q_\alpha^2\Big)\ E_\| \quad \overset{\text{with (9)}}{\Rightarrow} \quad \boxed{\sigma_H = \frac{e^2}{h}\sum_\alpha Q_\alpha^2,} \qquad (10)$$

where $Q_\alpha \cdot e$ is fractional electric charge of quasi-particle species α.

Edge- and bulk effective actions

Apparently, if $\sigma_H \notin \frac{e^2}{h}\mathbb{Z}$ then there exist fractionally charged quasi-particles propagating along supp$(\underline{\nabla}\sigma_H)$!
Chiral edge current d. J^μ_{edge} = generator of $U(1)$- current algebra (free massless fields!) Green functions of J^μ_{edge} obtained from 2D anomalous effective action $\Gamma_{\partial\Omega\times\mathbb{R}}(A_\|) = \cdots$, where $A_\|$ is restriction of vector potential, A, to boundary $\partial\Omega \times \mathbb{R}$.
Anomaly of $\sigma_H \Gamma_{\partial\Omega\times\mathbb{R}}(A_\|)$ – consequence of fact that J^μ_{edge} is *not* cons. – is cancelled by the one of bulk effective action, $S_{\Omega\times\mathbb{R}}(A)$:

$$\begin{aligned} j^\mu_{bulk}(x) &= \langle J^\mu(x)\rangle_A \equiv \frac{\delta S_{\Omega\times\mathbb{R}}(A)}{\delta A_\mu(x)} \\ &\overset{(6)!}{=} \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x), \quad x \notin \partial\Omega \times \mathbb{R} \end{aligned}$$

$$\Rightarrow \quad \boxed{S_{\Omega\times\mathbb{R}}(A) = \frac{\sigma_H}{2}\int_{\Omega\times\mathbb{R}} A \wedge \mathrm{d}A + \sigma_H \Gamma_{\partial\Omega\times\mathbb{R}}(A_\|)} \qquad (11)$$

Chern-Simons action on manifold with boundary!

Classification of "abelian" QH fluids (with help from L.M.)

Chiral anomaly (10) $\Rightarrow$ several (N) species of gapless quasi-particles propagating along edge $\leftrightarrow$ described by N chiral scalar Bose fields $\{\varphi^{\alpha}\}_{\alpha=1}^{N}$ with propagation speeds $\{v_\alpha\}_{\alpha=1}^{N}$, such that

1. Chiral electric edge current operator & Hall conductivity

$$J^{\mu}_{edge} = e \sum_{\alpha=1}^{N} Q_\alpha \, \partial^{\mu} \varphi^{\alpha}, \quad Q = (Q_1, \ldots, Q_N), \quad \sigma_H = \frac{e^2}{h} Q \cdot Q^T$$

2. Multi-electron/hole states loc. along edge created by vertex ops.

$$: \exp i \left(\sum_{\alpha=1}^{N} q^j_\alpha \varphi^{\alpha} \right) :, \; q^j = \begin{pmatrix} q^j_1 \\ \vdots \\ q^j_N \end{pmatrix} \in \Gamma, \; j = 1, \ldots, N. \quad (12)$$

Charge $\leftrightarrow$ Statistics $\Rightarrow$ Γ an odd-integral lattice of rank N. Hence:

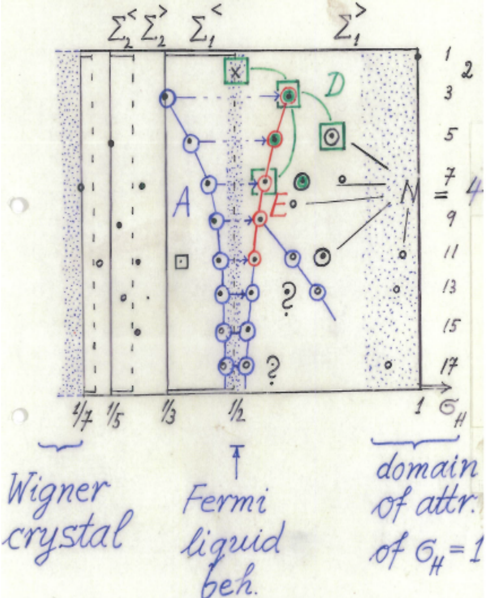
3. Classifying data are

$$\{ \Gamma;\, Q \in \Gamma^* : \text{“visible”};\, (q^j_\alpha)^N_{j,\alpha=1} : \sim \text{CKM matrix};\; v = (v_\alpha)^N_{\alpha=1} \}$$

$\rightarrow$ quasi-particles w. abelian braid statistics!

Success of classification – comparison with data

Γ = odd-integral lattice, $Q \in \Gamma^* \quad \Rightarrow \quad (\frac{e^2}{h})^{-1}\sigma_H \in \mathbb{Q}\,(!)\,, \ldots$



2. Chiral Spin Currents in Planar Topological Insulators

So far, we have not paid attention to electron spin, although there are 2D EG exhibiting the fractional quantum Hall effect where spin plays an important role. Won't study these systems, today. Instead, we consider time-reversal-invariant 2d topological insulators (2D TI) exhibiting chiral spin currents.

Pauli Eq. for a spinning electron:

$$i\hbar D_o \Psi_t = -\frac{\hbar^2}{2m} g^{-1/2} D_k \left(g^{1/2} g^{kl}\right) D_l \Psi_t, \tag{13}$$

where m is the mass of an electron, (g_{kl}) = metric of sample,

$$\Psi_t(x) = \begin{pmatrix} \psi_t^{\uparrow}(x) \\ \psi_t^{\downarrow}(x) \end{pmatrix} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 : \qquad \text{2-component Pauli spinor}$$

$$i\hbar D_0 = i\hbar\partial_t + e\varphi - \underbrace{\vec{W}_0 \cdot \vec{\sigma}}_{\text{Zeeman coupling}} \quad , \quad \vec{W}_0 = \mu c^2 \vec{B} + \frac{\hbar}{4} \vec{\nabla} \wedge \vec{V} \tag{14}$$

$U(1)_{em} \times SU(2)_{spin}$-gauge invariance

$$\frac{\hbar}{i} D_k = \frac{\hbar}{i} \partial_k + eA_k - m_0 V_k - \vec{W}_k \cdot \vec{\sigma}, \tag{15}$$

where $\vec{A}$ is em vector potential, $\vec{V}$ is velocity field describing mean motion (flow) of sample, $(\vec{\nabla} \cdot \vec{V} = 0)$,

$$\vec{W}_k \cdot \vec{\sigma} := \underbrace{[(-\tilde{\mu}\vec{E} + \frac{\hbar}{c^2}\dot{\vec{V}}) \wedge \vec{\sigma}]_k,}_{\text{spin-orbit interactions}}$$

and $\tilde{\mu} = \mu + \frac{e\hbar}{4mc^2}$ ($\leftarrow$ Thomas precession).
Note that the Pauli equation (13) respects $U(1)_{em} \times SU(2)_{spin}$ - gauge invariance.
We now consider an interacting 2D gas of electrons confined to a region Ω of the xy- plane, with $\vec{B} \perp \Omega$ and $\vec{E}, \vec{V} \| \Omega$. Then the $SU(2)$ - conn., $\vec{W}_\mu$, is given by $W^3_\mu \cdot \sigma_3$, $(W^M = 0,$ for $M = 1, 2)$.

Effective action of a 2D TI

Thus the connection for parallel transport of the component $\psi^{\uparrow}$ of Ψ is given by $a + w$, while parallel transport of $\psi^{\downarrow}$ is determined by $a - w$, where $a_\mu = -eA_\mu + mV_\mu, w_\mu = W^3_\mu$. These connections are abelian, (phase transformations). Under time reversal,

$$a_0 \to a_0, \ a_k \to -a_k, \text{ but } w_0 \to -w_0, \ w_k \to w_k. \tag{16}$$

The dominant term in the effective action of a 2D insulator is a Chern-Simons term. If there were only the gauge field a, with $w \equiv 0$, or only the gauge field w, with $a \equiv 0$, a Chern-Simons term would *not* be invariant under time reversal, and the dominant term would be given by

$$S(a) = \int \mathrm{d}t\mathrm{d}^2x \ \{\varepsilon \underline{E}^2 - \mu^{-1} B^2\} \tag{17}$$

But, in the presence of two gauge fields, a and w, satisfying (16):

Effective action of a 2D TI, ctd.

Combination of two Chern-Simons terms is time-reversal invariant:

$$
\begin{aligned}
S(a, w) &= \frac{\sigma}{2} \int \{(a + w) \wedge \mathrm{d}(a + w) - (a - w) \wedge \mathrm{d}(a - w)\} \\
&= \sigma \int \{a \wedge \mathrm{d}w + w \wedge \mathrm{d}a\}
\end{aligned}
$$

This reproduces (17) for phys. choice of w! ($\nearrow$ J.F., Les Houches '94!) – The gauge fields a and w transform independently under gauge transformations, and the Chern-Simons action is anomalous under these gauge trsfs. on a 2D sample space-time $\Lambda = \Omega \times \mathbb{R}$ with a non-empty boundary, $\partial\Lambda$. The anomalous chiral boundary actions,

$$\pm\sigma\Gamma\big((a \pm w)|_{\|}\big),$$

cancel anomaly of bulk action! Are generating functionals of conn. Green functions of two counter-propagating chiral edge currents:

Edge degrees of freedom: Spin currents

One of the two counter propagating edge currents has "spin-up" (in +3-direction, $\perp \Omega$), the other one has "spin down". Thus, a net chiral spin current, s^3_{edge}, can be excited to propagate along the edge; but there is no net electric edge current!

Response Equations, (2 oppositely (spin-)polarized bands):

$$\underline{j}(x) = 2\sigma(\underline{\nabla} B)^*, \qquad \text{and}$$

$$s_3^{\mu}(x) = \frac{\delta S(a, w)}{\delta w_{\mu}(x)} = 2\sigma \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x) \tag{18}$$

$\Rightarrow$ edge spin current – as in (7)!

We should ask what kinds of quasi-particles may produce the (bulk) Chern-Simons terms

$$S_{\pm}(a \pm w) = \pm \frac{\sigma}{2} \int \{(a \pm w) \wedge \, \mathrm{d}(a \pm w),$$

where, apparently $+$ stands for "spin-up" and $-$ stands for "spin-down". Well, it has been known ever since the seventies [1] that a two-component relativistic Dirac fermion with mass $M > 0$ ($M < 0$), coupled to an abelian gauge field A, breaks parity and time-reversal invariance and induces a Chern-Simons term

$$\underset{(-)}{+} \frac{1}{2\pi} \int A \wedge \, \mathrm{d} \, A$$

We thus argue that a 2D time-reversal invariant topological insulator with chiral edge spin-current exhibits two species of charged quasi-particles in the bulk, with one species (spin-up) related to the other one (spin-down) by time reversal, and each species has two degenerate states per wave vector mimicking a 2-component Dirac fermion (at small wave vectors).

[1] the first published account of this observation – originally due to Magnen, Sénéor and myself – appears in a paper by Deser, Jackiw and Templeton of 1982

Conclusions

• Physics in 2D is surprisingly rich. Important problems – in particular, ones concerning phase transitions and critical phenomena – appear to be exactly solved, using techniques ranging from the Bethe ansatz and the use of solutions to the Yang-Baxter equation, over 2D CFT, SLE, all the way to discrete-holomorphic functions. Yet, qualitative analysis, such as multi-scale analysis (K-T transition), still has a significant role to play.

• 2DEG, Bose gases and magnetic materials are fascinating play grounds for experimentalists and theorists alike, because general principles, such as anomalies and their cancellation, holography, two-comp. Dirac-like fermions, braid statistics, fractional spin & fractional electric charges, etc. all appear to manifest themselves in the physics of specific 2D systems.

• It is interesting to consider higher-dimensional cousins of the QHE and of topological insulators invariant under T. They are likely to be relevant in cosmology – in connection with the generation of primordial magnetic fields in the Universe, Dark Matter & Dark Energy. But these matters are left for another occasion.

Je vous remercie de votre attention!

“Survivre et Vivre” – 47 years later

… depuis fin juillet 1970 je consacre la plus grande partie de mon temps en militant pour le mouvement *Survivre*, fondé en juillet à Montréal. Son but est la lutte pour la survie de l’espèce humaine, et même de la vie tout court, menacée par le déséquilibre écologique croissant causé par une utilisation indiscriminée de la science et de la technologie et par des mécanismes sociaux suicidaires, et menacée également par des conflits militaires liés à la prolifération des appareils militaires et des industries d’armements … !

Alexandre Grothendieck