Critical BCS-temperature in a constant magnetic field

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We study the bottom of the *spectrum* of the following two-particle operator: This is the *second variation* of the corresponding BCS-functional.

$$M_{T,B} + V \; : \; L^2(\mathbb{R}^3 \times \mathbb{R}^3) \to L^2(\mathbb{R}^3 \times \mathbb{R}^3)$$

$$M_{T,B} = \frac{\mathfrak{h}_x + \mathfrak{h}_y}{\tanh \frac{\mathfrak{h}_x}{2T} + \tanh \frac{\mathfrak{h}_y}{2T}}, \qquad V = V(x-y)$$

$$\mathfrak{h}_x = \left(-i\nabla_x + \frac{\mathbf{B}}{2} \wedge x\right)^2 - \mu = \Pi_x^2 - \mu, \qquad \Pi_x = p_x + \frac{\mathbf{B}}{2} \wedge x$$

with μ chemical potential, T temperature, and

$$\mathbf{B} = (0, 0, B)$$

Goal: Obtain T as a function of B under the condition that

$$\inf \sigma(M_{T,B} + V) = 0.$$

In this way we want to obtain the critical temperature as a function of B, i.e., $T_c(B)$, for small B.

Idea

The *ground-state eigenfunction* separates into relative and center-of-mass motion

$$\alpha(x,y) \simeq \alpha_*(x-y)\psi\left(\frac{x+y}{2}\right) = \alpha_*(r)\psi(X),$$

$$r = x-y, \quad p_r = \frac{p_x - p_y}{2}, \qquad x = \frac{r}{2} + \frac{X}{2}, \quad p_x = p_r + \frac{p_X}{2}$$

$$X = \frac{x+y}{2}, \quad p_X = p_x + p_y, \qquad y = -\frac{r}{2} + \frac{X}{2}, \; p_y = -p_r + \frac{p_X}{2}$$

$$\Pi_x = p_x + \frac{\mathbf{B}}{2} \wedge x = p_r + \frac{p_X}{2} + \frac{\mathbf{B}}{4} \wedge r + \frac{\mathbf{B}}{2} \wedge X = \Pi_r + \frac{\Pi_X}{2}$$

$$\Pi_y = p_y + \frac{\mathbf{B}}{2} \wedge y = -p_r + \frac{p_X}{2} - \frac{\mathbf{B}}{4} \wedge r + \frac{\mathbf{B}}{2} \wedge X = -\Pi_r + \frac{\Pi_X}{2}$$

$$\langle \alpha, M_{T,B}\alpha \rangle = \left\langle \alpha, \frac{\left(\Pi_r + \frac{\Pi_X}{2}\right)^2 - \mu + \left(\Pi_r - \frac{\Pi_X}{2}\right)^2 - \mu}{\tanh \frac{\left(\Pi_r + \frac{\Pi_X}{2}\right)^2 - \mu}{2T} + \tanh \frac{\left(\Pi_r - \frac{\Pi_X}{2}\right)^2 - \mu}{2T}} \alpha \right\rangle$$

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$$\Pi_x = p_x + \frac{\mathbf{B}}{2} \wedge x = \left(p_r + \frac{\mathbf{B}}{4} \wedge r\right) + \left(\frac{p_X}{2} + \frac{\mathbf{B}}{2} \wedge X\right) = \Pi_r + \frac{\Pi_X}{2}$$

$$\Pi_y = p_y + \frac{\mathbf{B}}{2} \wedge y = -\left(p_r + \frac{\mathbf{B}}{4} \wedge r\right) + \left(\frac{p_X}{2} + \frac{\mathbf{B}}{2} \wedge X\right) = -\Pi_r + \frac{\Pi_X}{2}$$

$$\langle \alpha, M_{T,B}\alpha \rangle = \left\langle \alpha, \frac{\left(\Pi_r + \frac{\Pi_X}{2}\right)^2 - \mu + \left(\Pi_r - \frac{\Pi_X}{2}\right)^2 - \mu}{\tanh \frac{\left(\Pi_r + \frac{\Pi_X}{2}\right)^2 - \mu}{2T} + \tanh \frac{\left(\Pi_r - \frac{\Pi_X}{2}\right)^2 - \mu}{2T}} \alpha \right\rangle$$

Idea

$$\alpha \simeq \alpha_*(r)\psi(X)$$

$$\langle \alpha, M_{T,B}\alpha\rangle \simeq \langle \alpha_*, \frac{\Pi_r^2-\mu}{\tanh\frac{\Pi_r^2-\mu}{2T}}\alpha_*\rangle\|\psi\|_2^2 + \Lambda_0\langle\psi, \Pi_X^2\psi\rangle + o(B)$$

$$\begin{aligned} 0 &= \inf_{\|\alpha\|=1}\langle\alpha, (M_{T,B}+V)\alpha\rangle \\ &\simeq \inf_{\|\alpha_*\|=1}\langle\alpha_*, \left(\frac{p_r^2-\mu}{\tanh\frac{p_r^2-\mu}{2T}} + V(r)\right)\alpha_*\rangle + \Lambda_0 \inf_{\|\psi\|=1}\langle\psi, \Pi_X^2\psi\rangle + o(B) \\ &\simeq -\Lambda_2\frac{T_c(0)-T}{T_c(0)} + \Lambda_0 2B + o(B), \end{aligned} \tag{1}$$

hence

$$T(B) = T_c(B) = T_c(0)\left(1 - \frac{\Lambda_0}{\Lambda_2}2B\right) + o(B).$$

Difficulties

There are severe difficulties:

- $M_{T,B}$ is an *ugly* symbol.
- $\mathbf{B} \wedge x$ is an unbounded perturbation of an unbounded operator
- the components of $(-i\nabla + \frac{\mathbf{B}}{2} \wedge x)$ do *not* commute
- We need an operator inequality of the form

$$M_{T,B} + V \geq \frac{p_r^2 - \mu}{\tanh \frac{p_r^2 - \mu}{2T}} + V(r) + c\Pi_X^2 + o(B), \qquad c > 0$$

As a way out, we will deal with the Birman-Schwinger version:

$$V \geq 0 : (M_{T,B} - V)\alpha = 0 \Longleftrightarrow V^{1/2} \frac{1}{M_{T,B}} V^{1/2} \varphi = \varphi, \quad \varphi = V^{1/2}\alpha$$

hence we study the equation

$$0 = \inf \sigma \left(1 - V^{1/2} \frac{1}{M_{T,B}} V^{1/2} \right)$$

$B = 0$ and $T_c(0)$

In order to determine $T_c(0)$ one has to solve for T:

$$0 = \inf \sigma(M_{T,0} + V)$$

We abbreviate $M_T = M_{T,0}$ which can be represented as multiplication operator.

$$\widehat{M_T \alpha}(p,q) = \frac{p^2 - \mu + q^2 - \mu}{\tanh \frac{p^2-\mu}{2T} + \tanh \frac{q^2-\mu}{2T}} \hat{\alpha}(p,q)$$

One has the algebraic inequality

$$M_T(p,q) \geq \frac{1}{2} \left(\frac{p^2 - \mu}{\tanh \frac{p^2-\mu}{2T}} + \frac{q^2 - \mu}{\tanh \frac{q^2-\mu}{2T}} \right) \geq 2T,$$

since

$$\frac{x}{\tanh \frac{x}{2T}} \geq 2T.$$

The task to solve for the critical temperature T_c is non-trivial, even in the $B = 0$ case.

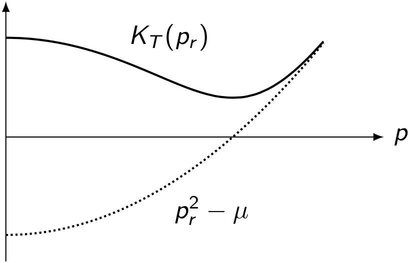
$M_T + V$ for $p_X = 0$ [HHSS]

At $p_X = 0$

$$M_T(p_r + \frac{p_X}{2}, -p_r + \frac{p_X}{2}) = M_T(p_r, -p_r)$$

$$= K_T(p_r) = \frac{p_r^2 - \mu}{\tanh((p_r^2 - \mu)/2T)}$$

$$K_T(-i\nabla) + V(r) : \ L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3).$$



Critical temperature: Since the operator $K_T + V$ is **monotone** in T, there exists unique $0 \leq T_c < \infty$ such that

$$\inf \sigma(K_{T_c} + V) = 0,$$

respectively 0 is the lowest eigenvalue of $K_{T_c} + V$.

T_c is the critical temperature for the *effective* one particle system ($p_X = 0$).

[HHSS] C. Hainzl, E. Hamza, R. Seiringer, J.P. Solovej, Commun. Math. Phys. **281**, 349–367 **(2008)**.

Known results about $K_T + V$

- $\lim_{T\to 0} \frac{p^2-\mu}{\tanh \frac{p^2-\mu}{2T}} = |p^2-\mu|$, hence

$$T_c > 0 \ \textbf{iff} \ \inf \sigma(|p^2-\mu| + V) < 0$$

- $\frac{1}{|p^2-\mu|}$ has same type of singularity as $1/p^2$ in $2D$ [S].
- In [FHNS, HS08, HS16] we classify V's such that $T_c > 0$. (E.g. $\int V < 0$ is enough)
- In [LSW] shown that $|p^2-\mu| + V$ has ∞ many eigenvalues if $V \leq 0$.
- the operator appeared in terms of scattering theory [BY93]

[FHNS] R. Frank, C. Hainzl, S. Naboko, R. Seiringer, Journal of Geometric Analysis, **17**, No 4, 549-567 (2007)

[HS08] C. Hainzl, R. Seiringer, Phys. Rev. B, **77**, 184517 (2008)

[HS16] C. Hainzl, R. Seiringer, J. Math. Phys. **57** (2016), no. 2, 021101

[BY93] Birman, Yafaev, St. Petersburg Math. J. 4, 1055-1079 (1993)

[LSW] A. Laptev, O. Safronov, T. Weidl, Nonlinear problems in mathematical physics and related topics I, pp. 233-246, Int. Math. Ser. (N.Y.), Kluwer/Plenum, New York (2002)

[S] B. Simon, Ann. Phys. 97, 279-288 (1976)

Lemma (FHSS12)

Let the 0 eigenvector of $K_{T_c} + V$ be non-degenerate. Then

(a)

$$M_{T_c} + V \gtrsim p_X^2$$

(b)

$$\inf \sigma(M_{T_c} + V) = 0$$

meaning $T_c(0)$ for the two-particle system is determined by T_c the critical temperature of the one-particle operator $K_T + V$, which satisfies $0 = \inf \sigma(K_{T_c} + V)$.

The proof of is non-trivial, because

$$M_T(p_r + \frac{p_X}{2}, -p_r + \frac{p_X}{2}) \not\gtrsim M_T(p_r, -p_r) = K_T(p_r).$$

(a) only holds for $V = V(x - y)$, *not* for general $V(x, y)$.

In the presence of **B** the proof is significantly harder and the main difficulty of our work.

[FHSS12] R.L. Frank, C. Hainzl, R. Seiringer, J.P. Solovej, J. Amer. Math. Soc. **25**, 667–713 (2012).

Main Theorem

$$0 = \inf \sigma \left(1 - V^{1/2} \frac{1}{M_{T,B}} V^{1/2} \right) \tag{2}$$

Theorem (FHL17)

Let $V \leq 0$, V and $|r|V(r)$ in L^∞, and the 0-eigenvector of $K_{T_c} + V$ be non-degenerate, then there are parameters Λ_0, Λ_2, depending on V, μ, such that there exists a solution $T_c(B)$ of equation (2) such that for small B

$$T_c(B) = T_c \left(1 - \frac{\Lambda_0}{\Lambda_2} 2B \right) + o(B),$$

This proves and generalizes a famous result of Helfand, Hohenberg and Werthamer [HHW].

[FHL17] R. L. Frank, C. Hainzl, E. Langmann, (2017)

[HHW] E. Helfand, Hohenberg, N.R. Werthamer, Phys. Rev. 147, 288 (1966)

Ingredients of the proof

Advantage: $M_{T,B}^{-1}$ can be expressed in terms of resolvents.

$$M_{T,B}^{-1} = \frac{1}{2i\pi} \int_C \tanh\left(\frac{z}{2T}\right) \frac{1}{z - \mathfrak{h}_x} \frac{1}{z + \mathfrak{h}_y} dz = T \sum_{n \in \mathbb{Z}} \frac{1}{\mathfrak{h}_x - i\omega_n} \frac{1}{\mathfrak{h}_y + i\omega_n}$$

with $\omega_n = \pi(2n+1)T$.

$$\begin{aligned}(M_{T,B}^{-1} V^{1/2}\alpha)(x,y) &= \\ \int\int dx' dy' \, T \sum_{n \in \mathbb{Z}} &\frac{1}{\mathfrak{h}_B - i\omega_n}(x,x') \frac{1}{\mathfrak{h}_B + i\omega_n}(y,y') V^{1/2}(x'-y')\alpha(x',y') \quad (3)\end{aligned}$$

We show

$$\frac{1}{z - \mathfrak{h}_B}(x,y) \simeq e^{-i\frac{\mathbf{B}}{2}\cdot x \wedge y} \frac{1}{z - \mathfrak{h}_0}(x-y)$$

and introduce center-of-mass and relative coordinates, and use

$$e^{-iZ\cdot(\mathbf{B}\wedge X)}\psi(X-Z) = e^{-iZ\cdot(\mathbf{B}\wedge X)}(e^{-iZ\cdot p_X}\psi)(X) = (e^{-iZ\cdot \Pi_X}\psi)(X)\,.$$

High-T_c-superconductors

But if V is more general, $V = V(r, X)$, then $M_T + V$ does not necessarily attain its *infimum* for $p_X = 0$ [HL].

This suggests a different type of pair formation in situations where the interaction is not translation-invariant.

We suggest that this happens in high-T_c-superconductors.

[HL] C. Hainzl, M. Loss, EPJ B (2017)

Previous results on the influence of A, W on critical temperature

In previous works we investigated the change of the critical temperature by bounded external fields Felder $h^2 W, h\mathbf{A}$ by means of the full *non-linear* functional. The shift in the critical temperature happens through the **lowest eigenvalue of the linearized Ginzburg-Landau operator**

$$D_c = \frac{1}{\Lambda_0} \inf \sigma \left(\Lambda_2(-i\nabla + 2\mathbf{A}(x))^2 + \Lambda_1 W(x) \right).$$

Theorem ([FHSS12, FHSS16])

If $\mathbf{A}$ and W are bounded and periodic, the ground state of $K_{T_c} + V$ non-degenerate, then there exist constants $\Lambda_0, \Lambda_1, \Lambda_2$ such that

$$T_c^{\mathrm{BCS}} = T_c(1 - D_c h^2) + o(h^2).$$

[FHSS16] R. L. Frank, C. Hainzl, R. Seiringer, J P Solovej, Commun. Math. Phys. 342 (2016), no. 1, 189–216
[FHSS12] R.L. Frank, C. Hainzl, R. Seiringer, J.P. Solovej, J. Amer. Math. Soc. **25**, 667–713 (2012).

Proof [FHSS12]

$$
\begin{aligned}
M_{T_c}(p_r + \frac{p_X}{2}, p_r - \frac{p_X}{2}) + V(r) &\geq \frac{1}{2}\left(K_{T_c}(p_r + \frac{p_X}{2}) + K_{T_c}(p_r - \frac{p_X}{2})\right) + V(r) \\
&= \frac{1}{2}\left(e^{ir\cdot p_X/2} K_{T_c}(p_r) e^{-ir\cdot p_X/2} + e^{-ir\cdot p_X/2} K_{T_c}(p_r) e^{ir\cdot p_X/2}\right) + V(r) \\
&= \frac{1}{2}\left(U_{p_X}[K_{T_c} + V]U_{p_X}^* + U_{p_X}^*[K_{T_c} + V]U_{p_X}\right) \\
&\geq \kappa\left(U_{p_X}[1 - |\alpha_*\rangle\langle\alpha_*|]U_{p_X}^* + U_{p_X}^*[1 - |\alpha_*\rangle\langle\alpha_*|]U_{p_X}\right) \\
&\geq \kappa\left[1 - \left|\int \cos(p_X \cdot r)|\alpha_*(r)|^2 dr\right|\right] \simeq c^2 p_X^2
\end{aligned}
$$

for small momenta p_X,

$$(K_{T_c} + V)\alpha_* = 0.$$

The proof crucially depends on $V = V(r)$ being translation invariant.

The proof is significantly harder if magnetic field **B** is included.

Using $e^{ir\cdot\Pi_X/2}$ has the difficulty that $r_1 \Pi_{X_1}$ and $r_2 \Pi_{X_2}$ do not commute.