Quantissima, Venice 2017

A nonlinear sigma model connected with stochastic processes and quantum diffusion.

Margherita DISERTORI

joint work with T.Spencer and M.Zirnbauer
C. Sabot and P. Tarrès
F. Merkl and S. Rolles

Bonn University & Hausdorff Center for Mathematics

I. A supersymmetric nonlinear sigma model $H^{2|2}$ (Zirnbauer 1996)

originally introduced as a toymodel for quantum diffusion

"spin" model:

$$d\mu(S)=\prod_{i\sim j\in\Lambda} e^{W_{ij}\langle S_i,S_j\rangle} \prod_{j\in\Lambda} e^{-\epsilon_j\langle e,S_j\rangle}\delta(\langle S_j,S_j\rangle+1) \ \frac{dS_\Lambda}{Z_\Lambda}$$

- $\Lambda\subset\mathbb{Z}^d$ finite cube, $i\sim j\equiv|i-j|=1$
- spin S is a (super)vector $S=(x,y,z,\xi,\eta)$,
  x,y,z even, ξ,η odd elements in a real Grassmann algebra
- $\langle S,S'\rangle=xx'+yy'-zz'+\xi\eta'-\eta\xi'$
- nonlinear constraint $\langle S,S\rangle=-1\Rightarrow z=\sqrt{1+x^2+y^2+2\xi\eta}$
- $W_{ij}>0,\ \langle S,S'\rangle\leq -1 \ \Rightarrow$ ferromagnetic interaction
- $\epsilon_j\geq 0$ mass term, $e=(0,0,1,0,0)$ $\langle e,S_j\rangle=(z_j-1)\geq 0$

Question: are the spins aligned as $\Lambda\to\mathbb{Z}^d$?

Horospherical coordinates: $(x,y,\xi,\eta)\to(u,s,\bar\psi,\psi)$

$d\mu(S)\to d\mu(u,s,\bar\psi,\psi)=$

$$\prod_{i\sim j\in\Lambda} e^{-W_{ij}(\cosh(u_i-u_j)-1)} \prod_{j\in\Lambda} e^{-\epsilon_j(\cosh u_j-1)} e^{-\frac{1}{2}(s,Ms)} e^{-(\bar\psi,M\psi)} d[u,s,\bar\psi,\psi]$$

$$d[u,s,\bar\psi,\psi]=\prod_{j\in\Lambda} du_j ds_j d\bar\psi_j d\psi_j e^{-u_j},$$

$$(s,Ms)=\sum_{i\sim j} W_{ij} e^{u_i+u_j}(s_i-s_j)^2+\sum_j \epsilon_j e^{u_j} s_j^2$$

main features

- zero mass: non-compact symmetry group $z^2-x^2-y^2-2\xi\eta=const$
- positive mass: compact symmetry subgroup (SUSY) $z^2=1+x^2+y^2+2\xi\eta=const$

$\Rightarrow$ normalized measure $Z=1$

- for this model a phase transition has been proved in $d\geq3$

[D.-Spencer-Zirnbauer 2010]

II. $H^{2|2}$ as a random walk in a random environment

The setting

- finite volume $\Lambda \subset \mathbb{Z}^d$, *undirected* edges $E_\Lambda = \{e = (j \sim k) \,|\, j,k \in \Lambda, |j-k| = 1\}$
- discrete time process: $(X_n)_{n \geq 0}$, $X_n \in \Lambda$
- time evolution: $n \to n+1$: conditional probability

$$\mathbb{P}(X_{n+1} = j | X_n = i, (X_k)_{k \leq n}) = \mathbf{1}_{i \sim j} \frac{\omega_{ij}(n)}{\sum_{k, k \sim i} \omega_{ik}(n)}$$

$\omega_{ij}(n) \geq 0$ local conductance at time n

- ▶ if ω independent of n: Markov chain in the environment ω
- ▶ if ω time dependent: memory effect

Question: is the process recurrent or transient as $\Lambda \to \mathbb{Z}^d$?

$(X_n)_{n\geq 0}$ is a random walk in a random environment (RWRE) if

$$\mathbb{P}_{0,\Lambda}[\cdot] = \int \mathbb{P}^{\omega}_{0\Lambda}[\cdot]\ d\rho_{0,\Lambda}(\omega)$$

$\mathbb{P}_{0,\Lambda}[\cdot]$ prob. for the process on Λ starting at 0

$\mathbb{P}^{\omega}_{0,\Lambda}[\cdot]$ Markov chain in a **frozen** environment $\omega=\{\omega_e\}_{e\in E_\Lambda}$

$d\rho_{0,\Lambda}(\omega)$ mixing measure

$H^{2|2}$ maps to the mixing measure of two RWRE models

- linearly edge-reinforced random walk ERRW (Diaconis 1986)
- vertex-reinforced jump process VRJP (Werner 2000, Volkov, Davis)

both processes tend to come back to sites already visited in the past (attractive interaction) *and are RWRE*

VRJP as RWRE: $\mathbb{P}^{\mathrm{VRJP}}_{0,W,\Lambda}[\cdot] = \int \mathbb{P}^{\omega(u,W)}_{0\Lambda}[\cdot] \; d\rho_{0,W,\Lambda}(u)$

- $u \in \mathbb{R}^{\Lambda}$ random vector with prob. law $d\rho_{0,W,\Lambda}(u)$
- $\omega_{ij}(u,W) = \frac{W_{ij}}{2} e^{u_j - u_i}$, $W_{ij} > 0$

connection with $H^{2|2}$:

$d\rho_{0,W,\Lambda}(u) =$ $u-$ marginal of $H^{2|2}$ with mass at 0: $\epsilon_j = \delta_{j0}\epsilon$

$$= \prod_{i\sim j} e^{-W_{ij}(\cosh(u_i - u_j) - 1)} \; e^{-\epsilon(\cosh u_0 - 1)} \sqrt{\det M(u)} \; du$$

$$du = \prod_j du_j e^{-u_j} (2\pi)^{-1/2},$$

ERRW as a RWRE: $\mathbb{P}^{\mathrm{ERRW}}_{0,a,\Lambda}[\cdot] = \int \mathbb{P}^{\mathrm{VRJP}}_{0,W,\Lambda}[\cdot] \; d\gamma_{0,a,\Lambda}(W)$

$\{W_e\}_{e\in E}$ independent gamma distributed r.v.:

$$d\gamma_{0,a,\Lambda}(W) \propto \prod_e e^{-W_e} W_e^{a_e - 1} dW_e \qquad a_e > 0$$

from mixing measure to localization/transience

remember: $\mathbb{P}_{0,\Lambda}[\cdot] = \int \mathbb{P}^{\omega}_{0\Lambda}[\cdot]\ d\rho_{0,\Lambda}(\omega)$

Two possible criterions:

e_0 a fixed arbitrary edge attached to 0

- **positive recurrence:** $\int d\rho_{0,\Lambda}(\omega)\Big[\big(\omega_e/\omega_{e_0}\big)^s\Big] \leq K\ e^{-c|e-e_0|}$
  unif. in Λ for some $0<s\leq 1$, $c,K>0$.

  analog to $\mathbb{E}_H\Big[\big|(E-H)^{-1}_{jk}\big|^s\Big]\leq e^{-c|j-k|}$ in quantum diffusion

- **transience** $(d \geq 3)$**:** $\int d\rho_{0,\Lambda}(\omega)\Big[(\omega_{e_0}/\omega_e)\Big] \leq K$
  unif. in Λ for some $K>0$.

  analog to $\mathbb{E}_H\Big[\big|(E+i\epsilon-H)^{-1}_{jj}\big|^2\Big]\leq K\ \ \epsilon=|\Lambda|^{-1}$ in quantum diffusion

Some results

positive recurrence

- strong reinf.: ERRW and VRJP **for any** $d \geq 1$

  [Merkl-Rolles 2009], [D.-Spencer 2010] [Sabot-Tarrès 2013]

  [Angel-Crawford-Kozma 2014]

- any reinf.: ERRW and VRJP in $d = 1$ and strips

  [Merkl-Rolles 2009], [D.-Spencer 2010] [Sabot-Tarrès 2013] [D.-Merkl-Rolles 2014]

transience

- weak reinf.: ERRW and VRJP in $d \geq 3$

  [D.-Spencer-Zirnbauer 2010], [D.-Sabot-Tarrès 2015]

$\Rightarrow$ phase transition in $d \geq 3$

key tool: Ward identities inherited by the supersymmetric structure of the $H^{2|2}$ measure

III. $H^{2|2}$ as a Random Schrödinger operator

$M(u)$ *as a RS matrix:* $e^{-u}M(u)e^{-u}=H_{\beta(u)}=2\beta(u)-P^W$

Laplacian: $P^W_{ij}=\mathbf{1}_{i\sim j}W_{ij}$, diagonal disorder: $\beta_{ij}=\mathbf{1}_{i=j}\beta_i$

$$\beta_j(u):=e^{-2u_j}M_{jj} = \epsilon_j e^{-u_j}+\sum_{k\sim j} e^{u_k-u_j}W_{jk}$$

law of the random potential: [Sabot-Tarrès-Zeng 2015]

$$d\nu_{\epsilon,W,\Lambda}(\beta)=\frac{1}{Z}\mathbf{1}_{H_\beta>0}\frac{1}{(\det H_\beta)^{1/2}}e^{-\frac{1}{2}(\epsilon,H_\beta^{-1}\epsilon)}e^{-\sum_j \beta_j}\prod_j d\beta_j$$

$$Z_{\epsilon,W,\Lambda}=(\tfrac{\pi}{2})^{|V|/2}e^{-\sum_{e\in E}W_e}e^{-\sum_j \epsilon_j}$$

properties of $d\nu_{\epsilon,W,\Lambda}(\beta)$*:* explicit formula for the Laplace transform

$$\int e^{-\sum_j \lambda_j\beta_j} d\nu_{\epsilon,W,\Lambda}(\beta) = \prod_j \frac{e^{\epsilon_j(1-\sqrt{1+\lambda_j})}}{\sqrt{1+\lambda_j}} \prod_{i\sim j} e^{W_{ij}(1-\sqrt{1+\lambda_j}\sqrt{1+\lambda_k})}$$

$$\lambda_j \geq -1$$

Some consequences: [Sabot-Tarrès-Zeng 2015] [D.,Merkl-Rolles 2016]

- the r.v. β_j, β_k are independent if $|j-k| \geq 2$.
- wired boundary conditions $\epsilon_i^\Lambda = \sum_{j\in\Lambda^c, j\sim i} W_{ij}$
  i.e. zero boundary conditions for u^Λ variables: $u_j^\Lambda = 0 \ \forall \ j \in \Lambda^c$
  - $\exists$ a unique prob. measure on $\mathbb{R}^{\mathbb{Z}^d}$ $d\nu_{\epsilon,W}^\infty(\beta)$ with marginals $d\nu_{\epsilon,W,\Lambda}(\beta)$ (Kolmogorov ext. th.)
  - martingale: set $\psi_j^\Lambda = e^{u_j^\Lambda}$, $j \in \mathbb{Z}^d$: $\forall \Lambda \subset \Lambda'$ $\quad \mathbb{E}[\psi^{\Lambda'} | \mathcal{F}_\Lambda] = \psi^\Lambda$.

Some other results

- ERRW in $d = 2$ recurrent for any reinforcement

[Merkl-Rolles 2009], [Sabot-Zeng 2015]

- infinite hierarchy of martingales with generating function:

$$M^{\Lambda}(\theta)=e^{(u^{\Lambda},\theta)}e^{-\frac{1}{2}(\theta,H_{\Lambda}^{-1}\theta)}, \qquad \theta\in(-\infty,0]^{\mathbb{Z}^d}$$

[D.,Merkl-Rolles 2016]

Some open problems

- $d = 2$ positive recurrence (exponential localization) is expected for both VRJP and ERRW
- $d \geq 3$ complete phase diagram? At the moment results only for very small/large reinf
- spectral properties for the RS H_β

THANK-YOU!