# Quantization of Hall conductance in gapped systems

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# Motivation: Two recent papers

Hastings and Michalakis (2015)

- Spin systems on discrete 2-torus
- Assume unique ground state with spectral gap
- Conserved local 'charge' $Q_x \to$ current and potential
- Result: Hall conductance is $(2\pi\times)$ integer.
- Tools: quasi-adiabatic flow $\to$ Talk of Bruno
- Hard to understand

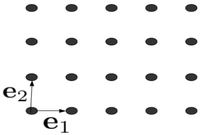
Giuliani, Mastropietro, Porta (2016)

- Weakly interacting fermions on discrete 2-torus
- Assume only that non-interacting system has spectral gap.
- Result: Hall conductance is $(2\pi\times)$ integer.
- Tool: Fermionic PT and Ward identities

Our goal: Simple rendering of (weakened) H-M, no original result

# Interacting fermions on 2-torus

- Discrete torus $(\mathbb{Z}/L\mathbb{Z})^2$ with sites $x$ and linear size $L$.




- Typical Hamiltonian

$$H = \underbrace{\sum_x (v_x - \mu) n_x + \sum_{x,i} w_i n_x n_{x+e_i}}_{=:D \text{ (diagonal in } n_x)} + \sum_{x,i} (\alpha_i c_x^* c_{x+e_i} + hc)$$

with $\{c_x, c_y^*\} = \delta_{x,y}$ and $n_x = c_x^* c_x$.

- Set local charge $Q_x \equiv n_x$.
- Unitary gauge transf. $V_\theta = \otimes_x \mathrm{e}^{-\mathrm{i}\theta(x) Q_x}$ for functions $\theta(x)$.

# Vector potential a

- Gauge transformation $V_\theta$ affects hopping

$$V_\theta H V_\theta^* = D + \sum_{x,i} (\alpha_i c_x^* c_{x+e_i} \mathrm{e}^{\mathrm{i}\nabla_i \theta(x)} + hc)$$

  with vector potential $a_i(x) = \nabla_i \theta(x) = \theta(x+e_i) - \theta(x)$.

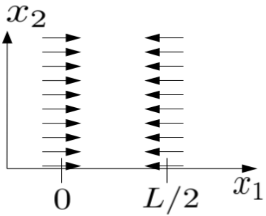
- For general fields $\mathbf{a} = \mathbf{a}(x)$

$$H^{\mathbf{a}} \equiv D + + \sum_{x,i} (\alpha_i c_x^* c_{x+e_i} \mathrm{e}^{\mathrm{i} a_i(x)} + hc)$$

  expect that $H^{\mathbf{a}} \neq V_\theta H V_\theta^*$ for some gauge $\theta$.

- We need just small class of $\mathbf{a}$: no $\mathbf{B}$ piercing the lattice, only thread fluxes through torus.

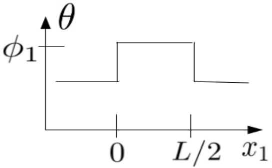
We define Twist-antitwist Hamiltonians $H(\phi_1, \phi_2)$:
Consider **a** inducing a twist $\phi_1$ and antitwist $-\phi_1$.

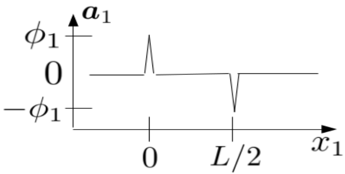



Pure gauge!

No net flux inserted

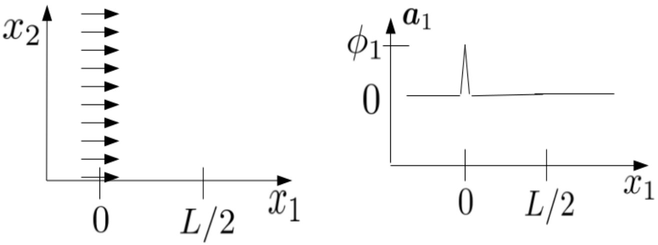
Or, flux inserted and
extracted
(==> Twist-antitwist)




Call resulting Hamiltonian $H(\phi_1) \equiv H^{\mathbf{a}} = V(\theta) H V^*(\theta)$.
Analagously, put also T-AT in 2-direction $\Rightarrow$ $H(\phi_1, \phi_2)$

We define Twist Hamiltonians $\tilde{H}(\phi_1, \phi_2)$:
Consider **a** inducing a twist flux $\phi_1$.




**Not pure gauge!** Net flux $\phi_1$ inserted

Call resulting Hamiltonian $\tilde{H}(\phi_1) = H^{\mathbf{a}}$. Analagously, put also T in 2-direction $\Rightarrow$ $\tilde{H}(\phi_1, \phi_2)$

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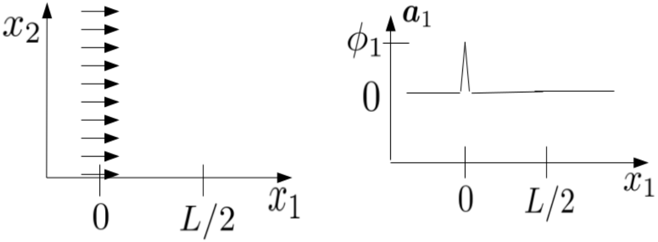


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Call resulting Hamiltonian $\tilde{H}(\phi_1) = H^{\mathbf{a}}$. Analagously, put also T in 2-direction $\Rightarrow \tilde{H}(\phi_1, \phi_2)$

- No obvious spectral relation between the $\tilde{H}(\phi_1, \phi_2)$.
- We write $H(\phi), \tilde{H}(\phi)$ with $\phi = (\phi_1, \phi_2)$.
- Fundamental objects will be $\tilde{H}(\phi)$ rather than $H(\phi)$.

# Torus $\mathbb{T}^2$ of fluxes $\phi = (\phi_1, \phi_2)$

Assumption: Family $\tilde{H}(\phi)$ has uniform gap (in $L$ and in $\phi$). Let $\tilde{P}(\theta)$ be the (rank-1) GS projection of $\tilde{H}(\theta)$.

**Fact 1: Hall Conductance = Berry curvature**

Hall conductance of $\tilde{H} = \tilde{H}(\phi)$ is given by ($\lim_{L\to\infty}(\cdot)$ of )

$$\kappa(\theta) = \mathrm{i}\,\mathrm{Tr}\,\tilde{P}[\partial_1\tilde{P}, \partial_2\tilde{P}], \qquad \partial_i = \partial_{\phi_i}$$

**Fact 2: Integral of Berry curvature = Chern number**

$$\frac{1}{2\pi}\int_{\mathbb{T}^2} d^2\theta\,\kappa(\theta) \text{ is an integer}$$

To conclude that Hall conductance is quantized, it hence suffices to show that $\kappa(\phi)$ is constant in $\phi$, as $L \to \infty$:

*'To remove averaging assumption'*

This is what I will mainly explain.

# Result and comments

**Theorem: $\kappa(\phi)$ constant**

1. $\sup|\kappa(\phi)-\kappa(\phi')| = \mathcal{O}(L^{-\infty})$ hence $d(\kappa(\phi), 2\pi\mathbb{Z}) = \mathcal{O}(L^{-\infty})$.
2. If TL limit exists: $\lim_L \mathrm{Tr}(P_L A)$ exists for any local $A$, then $(1/2\pi)\lim_L \kappa_L(\phi)$ exists and is integer.

- Setup: Spin systems, finite rangle, locally conserved charges $Q_x$ with integer spectrum. $\Rightarrow$ straightforward definition of fluxes, potentials. . . .
- Lattice fermions also OK by forthcoming work of Nachtergaele-Sims-Young.
- Gap assumption for weakly interacting fermions: proof by fermionic cluster expansion (Salmhofer, in preparation)
- Gap assumption in general. Perhaps intuitive argument that gap at $\phi = 0$, then gap at $\phi \neq 0$.

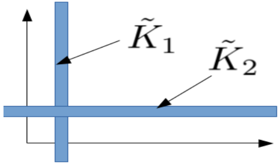
1. Local Generator of evolution in $\theta$ (Bruno's talk)

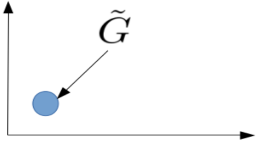
$$\partial_i \tilde{P} = -\mathrm{i}[\tilde{K}_i, \tilde{P}], \qquad i = 1, 2$$

   $\tilde{K}_i$ can be chosen as *(quasi-)local Hamiltonians*, unlike $\mathrm{i}[P, \partial_i P]$

2. Local perturbations perturb locally $\tilde{K}_i$ acts only where the perturbing field **a** is nonzero.
3. Recast $\kappa$ using $\tilde{P}\tilde{K}_i\tilde{P} = 0$

$$\kappa = \mathrm{i}\,\mathrm{Tr}\,\tilde{P}[\partial_1\tilde{P}, \partial_2\tilde{P}] = \mathrm{Tr}\,\tilde{P}G, \qquad \text{with } \tilde{G} = \mathrm{i}[\tilde{K}_1, \tilde{K}_2]$$



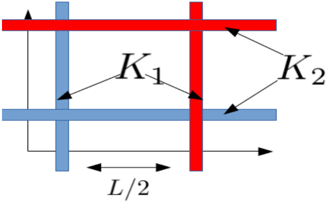



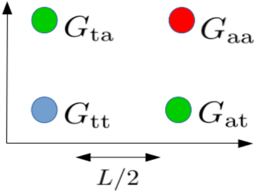
Same applies for generators $K_i$ implementing the twist-antiwists.

- There are local Hamiltonians $K_i$

$$\partial_i P = -\mathrm{i}[K_i, P], \qquad i = 1, 2$$

- Now $\mathrm{i}[K_1, K_2] = G = G_{\mathrm{tt}} + G_{\mathrm{ta}} + G_{\mathrm{at}} + G_{\mathrm{aa}}$




- But, twist-antitwist are pure gauge $\Rightarrow$ each of the quantities $A = P, K_i, G$ is given by

$$A(\phi) = V_\theta A(0) V_\theta^*, \qquad \text{for some gauge } \theta = \theta(\phi)$$

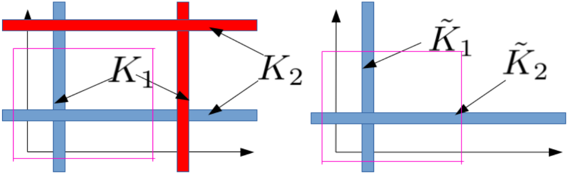
Since $V_\theta$ acts locally and $G$ is sum of distant terms, also

$$G_{tt}(\phi) = V_\theta G_{tt}(0) V_\theta^* \qquad (\text{up to } \mathcal{O}(L^{-\infty}))$$

# Locally, Twist = Twist-Antitwist

Generators $K_i, \tilde{K}_i$ depend locally on the $H, \tilde{H}$, so

$$K_i = \tilde{K}_i \quad \text{in the pink box}$$




Generators $K_i, \tilde{K}_i$ generate the $P, \tilde{P}$, so also

$$\mathrm{Tr}(PO) = \mathrm{Tr}(\tilde{P}O) \quad \text{for } O \text{ in the pink box}$$

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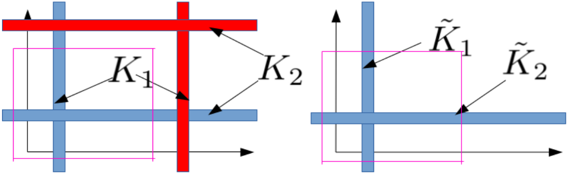


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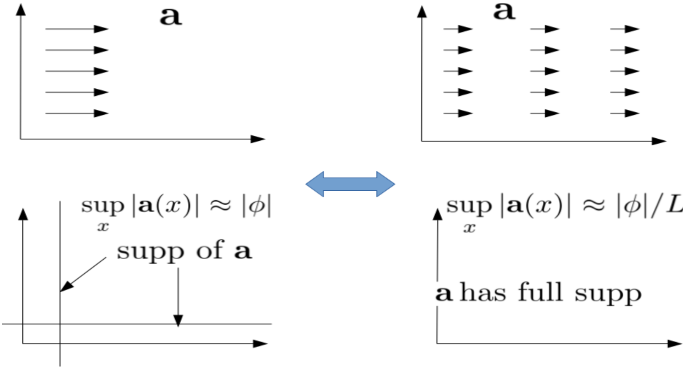
Now we are done:

$$\kappa = \mathrm{Tr}\,\tilde{P}\tilde{G} = \mathrm{Tr}\,\tilde{P}G_{\mathrm{tt}} = \mathrm{Tr}\,PG_{\mathrm{tt}}$$

Since $PG_{\mathrm{tt}}$ depends on $\phi$ unitarily, its trace is $\phi$-independent, hence so is $\kappa$

# Comment on gap assumption

By unitary gauge trafo "spread vector potentials over full volume

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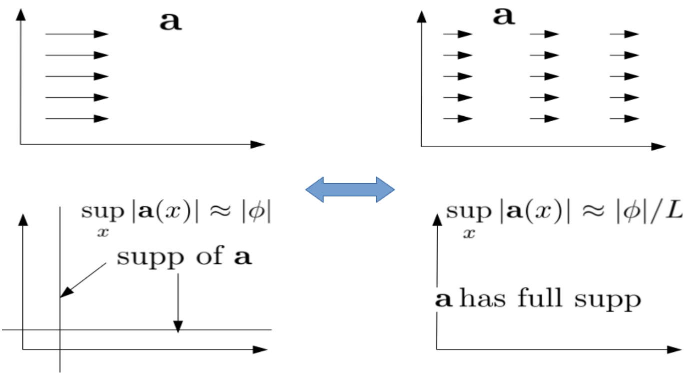


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Anyhow, Hastings-Michalakis need gap assumption only for *small* $\phi$. More reason for this to hold than for any $\phi$?