Spectral properties of Bose gases interacting through singular potentials

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based on joint works with
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The dilute Bose gas in 3d

We consider N bosons enclosed in a cubic box Λ of side length L, described by

$$H_N = -\sum_{j=1}^{N} \Delta_{x_j} + \sum_{1\le i<j\le N} V(x_i - x_j)\,, \quad \rho a^3 \ll 1$$

Long-standing goals: for $N \to \infty$ and $\rho = N/|\Lambda|$ fixed

- **prove the occurrence of condensation**
- **ground state energy and the low lying excitation spectrum**

Expected: Bogoliubov theory, Lee-Huang-Yang formula

$$\lim_{\substack{N\to\infty \\ \rho=\text{const.}}} \frac{E_N}{N} = 4\pi\rho a\left[1 + \frac{128}{15\pi}(\rho a^3)^{1/2} + o(\rho a^3)^{1/2}\right]$$

Results: condensation: hard-core bosons at half filling [Dyson-Lieb-Simon,'78]
ground state energy at leading order [Dyson'57], [Lieb-Yngvason,'98]
LHY for high density regime [Giuliani-Seiringer '09]
2^{nd} order upper bound [Erdös-Schlein-Yau,'08], [Yau-Yin, '13]
RG [Benfatto '94], [Balaban-Feldman-Knörrer-Trubowitz '08-'16]

Open: condensation? 2^{nd} order in other regimes? spectrum?

Mean field regime and Bogoliubov theory

N bosons in $\Lambda = [0;1]^{\times 3}$, periodic boundary conditions, **mean field regime**

$$H_N^{mf} = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{1}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r) a_{p+r}^* a_q^* a_p a_{q+r} \qquad \Lambda_* = 2\pi\mathbb{Z}^3$$

Bogoliubov approximation has been proved to be valid [Seiringer '11]

- Condensation with rate of convergence: $1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq CN^{-1}$
- Ground state energy at second order

$$E_N^{mf} = \frac{(N-1)\widehat{V}(0)}{2} - \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[p^2 + \kappa \widehat{V}(p) - \sqrt{|p|^4 + 2\kappa |p|^2 \widehat{V}(p)} \right] + o(1)$$

- Bogoliubov spectrum of elementary excitations

$$\sum_{p \in \Lambda_+^*} n_p \sqrt{|p|^4 + 2\kappa p^2 \widehat{V}(p)} + o(1), \quad n_p \in \mathbb{N}$$

Further results: [Grech-Seiringer '13], [Lewin-Nam-Serfaty-Solovey '14], [Derezinski-Napiorkovski '14], [Pizzo '15]

The Gross-Pitaevskii regime

Strong and short range interactions among atoms in BEC experiments can be modelled by the **Gross-Pitaevskii potential**:

$$H_N^{GP} = \sum_{i=1}^{N} \left(-\Delta_{x_i} + V_{ext}(x_j) \right) + \sum_{i<j}^{N} N^2 V\left(N(x_i - x_j)\right)$$

- If $V(x)$ has scattering length a_0, then $N^2 V(Nx)$ has scattering length $a = a_0/N$
- dilute regime $\rho a^3 = N^{-2}$
- **correlations among the particles** play a crucial role to understand both statical and dynamical properties of the system

Gross-Pitaevskii regime: ground state properties

N bosons in $\Lambda = [0;1]^{\times 3}$, periodic boundary conditions

$$H_N^{GP} = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{1}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N)\, a_{p+r}^* a_{q-r}^* a_p a_q$$

From [Lieb-Seiringer-Yngvason, '00] the ground state energy of H_N^{GP} at leading order is

$$E_N = 4\pi a_0 N + o(N)$$

From [Lieb-Seiringer, '02] the one particle reduced density $\gamma_N^{(1)}$ associated to the ground state of H_N^{GP} is such that in trace norm

$$\gamma_N^{(1)} \xrightarrow[N\to\infty]{} |\varphi_0\rangle\langle\varphi_0|$$

where $\varphi_0(x) = 1$ for all $x \in \Lambda$.

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$$E_N = 4\pi a_0 N + o(N) \qquad 8\pi a_0 = \int \mathrm{d}x\, f(x) V(x)$$

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Next order?
Low energy states?

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Strong BEC bounds?

where $\varphi_0(x) = 1$ for all $x \in \Lambda$.

Condensation in the Gross-Pitaevskii regime

N bosons in $\Lambda = [0;1]^{\times 3}$, periodic boundary conditions

$$H_N^{GP} = \sum_{p \in \Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N} \sum_{p,q,r \in \Lambda^*} \widehat{V}(r/N)\, a_{p+r}^* a_{q-r}^* a_p a_q$$

[Boccato-Brennecke-C.-Schlein '17] Let $V \in L^3(\mathbb{R}^3)$ be non-negative, spherically symmetric and compactly supported and suppose $\kappa \geq 0$ to be small enough. Let $\psi_N \in L_s^2(\Lambda^N)$ be a sequence with $\|\psi_N\| = 1$ and

$$\langle \psi_N, H_N \psi_N \rangle \leq 4\pi a_0 N + K$$

for some $K > 0$. Let $\gamma_N^{(1)}$ the one-particle reduced density associated with ψ_N. Then there exists $C > 0$ such that

$$N(1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle) \leq C(K+1)$$

with $\varphi_0(x) = 1$ for all $x \in \Lambda$.

Remark. The theorem applies to the g.s. & all states with excitation energy $\ll N$

Spectral properties for singular potentials

$$H_N^\beta = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^\beta)\, a_{p+r}^* a_{q-r}^* a_p a_q \qquad \beta \in (0;1)$$

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$$E_N^\beta = 4\pi a_N^\beta (N-1) - \frac{1}{2} \sum_{p\in\Lambda_+^*} \left[p^2 + \kappa\widehat{V}(0) - \sqrt{|p|^4 + 2\kappa p^2 \widehat{V}(0)} - \frac{\kappa^2 \widehat{V}^2(0)}{2p^2} \right] + o(1)$$

where $\Lambda_+^* = 2\pi\mathbb{Z}^3 \setminus \{0\}$ and

$8\pi a_N^\beta$

with $m_\beta \in \mathbb{N}$ the largest integer with $m_\beta \leq 1/(1-\beta) + \min(1/2, \beta/(1-\beta))$.

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where $\Lambda_+^* = 2\pi\mathbb{Z}^3 \setminus \{0\}$ and

$$8\pi a_N^\beta = k\widehat{V}(0) - \frac{1}{2N} \sum_{p\in\Lambda_+^*} \frac{\kappa^2 \widehat{V}^2(p/N^\beta)}{2p^2}$$

$$+ \sum_{m=2}^{m_\beta} \frac{(-1)^m \kappa^m}{(2N)^m} \sum_{p_1,\dots,p_m\in\Lambda_+^*} \frac{\widehat{V}(p_1/N^\beta)}{p_1^2} \left[\prod_{j=1}^{m-1} \frac{\widehat{V}((p_j - p_{j+1})/N^\beta}{p_{j+1}^2} \right] \widehat{V}(p_m/N^\beta)$$

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$$+\sum_{m=2}^{m_\beta}\frac{(-1)^m\kappa^m}{(2N)^m}\sum_{p_1,\dots,p_m\in\Lambda_+^*}\frac{\widehat{V}(p_1/N^\beta)}{p_1^2}\left[\prod_{j=1}^{m-1}\frac{\widehat{V}((p_j-p_{j+1})/N^\beta)}{p_{j+1}^2}\right]\widehat{V}(p_m/N^\beta)$$

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$$8\pi \tilde{a}_N^\beta = \kappa\widehat{V}(0) - \sum_{p\in\Lambda_+^*} \kappa \widehat{V}(p/N^\beta)\, \widehat{\omega}_N(p)$$

with $m_\beta \in \mathbb{N}$ the largest integer with $m_\beta \le 1/(1-\beta) + \min(1/2, \beta/(1-\beta))$.
Moreover, the spectrum of $H_N^{(\beta)} - E_N^\beta$ below ζ consists of

$$\sum_{p\in\Lambda_+^*} n_p \sqrt{|p|^4 + 2\kappa p^2 \widehat{V}(0)} + o(1)\,, \quad \text{with } n_p \in \mathbb{N}$$

Orthogonal excitations [Lewin-Nam-Serfaty-Solovej '12]

$$H_N^\beta = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^\beta) a_{p+r}^* a_{q-r}^* a_p a_q \quad \beta \in [0,1]$$

For $\psi_N \in L_s^2(\Lambda^N)$ and $\varphi_0 \in L^2(\Lambda)$

$$\psi_N = \alpha_0\, \varphi_0^{\otimes N} + \alpha_1 \otimes_s \varphi_0^{\otimes N-1} + \ldots + \alpha_{N-1} \otimes_s \varphi_0 + \alpha_N\,,$$

where $\alpha_j \in L^2(\Lambda)^{\otimes_s j}$ and $\alpha_j \perp \varphi_0$.

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Unitary map $U_{\varphi_0} : L_s^2(\Lambda^N) \longrightarrow \mathcal{F}_{\perp\varphi_0}^{\leq N} = \bigoplus_{n=0}^N L^2_{\perp\varphi_0}(\Lambda)^{\otimes_s n}$

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The excitation Hamiltonian

Conjugation with U remind c-number substitution in Bogoliubov approximation

$$U\, a_0^* a_0\, U^* = N - \mathcal{N}_+ \qquad U\, a_0^* a_p\, U^* = \sqrt{N - \mathcal{N}_+}\, a_p$$

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We define $\mathcal{L}_N^\beta = U H_N^\beta U^* : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$

$$\begin{aligned}\mathcal{L}_N^\beta = &\frac{N-1}{2N} \kappa \widehat{V}(0)(N - \mathcal{N}_+) + \frac{\kappa \widehat{V}(0)}{2N} \mathcal{N}_+ (N - \mathcal{N}_+) + \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p \\ &+ \sum_{p \in \Lambda_+^*} \kappa \widehat{V}(p/N^\beta) \left[b_p^* b_p - \tfrac{1}{N} a_p^* a_p \right] + \frac{\kappa}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N^\beta) \left[b_p^* b_{-p}^* + b_p b_{-p} \right] \\ &+ \frac{\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N^\beta) \left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \\ &+ \frac{\kappa}{2N} \sum_{p,q \in \Lambda_+^*, r \in \Lambda^* : r \neq -p,-q} \widehat{V}(r/N^\beta) a_{p+r}^* a_q^* a_p a_{q+r}\end{aligned}$$

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& + \sum_{p \in \Lambda_+^*} \kappa \widehat{V}(p/N^\beta) \left[b_p^* b_p - \tfrac{1}{N} a_p^* a_p \right] + \frac{\kappa}{2} \sum_{p \in \Lambda_+^*} \widehat{V}(p/N^\beta) \left[b_p^* b_{-p}^* + b_p b_{-p} \right] \\
& + \frac{\kappa}{\sqrt{N}} \sum_{p,q \in \Lambda_+^* : p+q \neq 0} \widehat{V}(p/N^\beta) \left[b_{p+q}^* a_{-p}^* a_q + a_q^* a_{-p} b_{p+q} \right] \\
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\end{aligned}$$

Remark. In the mean-field regime the expected value of cubic and quartic terms vanishes, as $N \to \infty$ when we consider low energy states.

Correlation structure

$H_N^\beta = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^\beta) a_{p+r}^* a_{q-r}^* a_p a_q$

For $\beta > 0$ some of the quartic terms in $\mathcal{L}_N^\beta = U\, H_N^\beta\, U^*$ are now important in the limit $N \to \infty$

Not surprising: the difference between the energy of a factorized state and the real ground state energy is of the order N^β

$$\langle \Omega, \mathcal{L}_N^\beta \Omega \rangle = \langle U^* \Omega, H_N^\beta U\, \Omega \rangle = \langle \varphi_0^{\otimes N} H_N^\beta \varphi_0^{\otimes N} \rangle = \frac{(N-1)\widehat{V}(0)}{2} \gg 4\pi a_N^\beta N$$

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States with small energy in the Gross-Pitaevskii limit and in the intermediate regimes $0 < \beta < 1$ are characterized by a correlation structure, which we model by the solution of the Neumann problem

$$\Big(-\Delta + \tfrac{k}{2} N^{3\beta-1} V(N^\beta x) \Big) f_{N,\ell}(x) = \lambda_{N,\ell} f_{N,\ell}(x)$$

on the ball $|x| \leq \ell < 1/2$, with

$$f_{N,\ell}(x) = 1 \quad \text{and} \quad \partial_{|x|} f_{N,\ell}(x) = 0 \qquad \text{for } |x| = \ell$$

Generalized Bogoliubov transformation

Inspired by the dynamics [Brennecke-Schlein '17] we describe correlations in $\mathcal{F}_+^{\leq N}$ using

$$T = \exp\Big[\,\frac{1}{2}\sum_{p\in\Lambda_+^*}\eta_p\big(b_p^* b_{-p}^* - b_p b_{-p}\big)\Big] : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$$

with $\quad \eta_p = -N\,\widehat{(1-f_{N,\ell})}(p) \qquad \Big(\eta_p \simeq \frac{\kappa}{p^2}\,,\ \sum_{p\in\Lambda_+^*} p^2|\eta_p|^2 \simeq \kappa^2 N^\beta\Big)$

and modified creation and annihilation operators

$$b_p^* = a_p^*\sqrt{\frac{N-\mathcal{N}}{N}}\,, \qquad b_p = \sqrt{\frac{N-\mathcal{N}}{N}}\,a_p$$

Remark. We can interpret b_p^* as an operator exciting a particle from the condensate to its orthogonal complement while b_p annihilates an excitation back into the condensate:

$$U^* b_p^* U = a_p^*\frac{a_0}{\sqrt{N}} \qquad U^* b_p U = \frac{a_0^*}{\sqrt{N}}a_p$$

Bounds on the modified excitation Hamiltonian

Define $\mathcal{G}_N = T^* U H_N U^* T : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, then

$$\mathcal{G}_N = 4\pi a_0 N + \mathcal{H}_N + \mathcal{E}_N$$

where

$$\mathcal{H}_N = \sum_{p \in \Lambda_+^*} p^2 a_p^* a_p + \frac{1}{2N} \sum_{\substack{p,q \in \Lambda_+^* \\ r \in \Lambda^* : r \neq -p-q}} \widehat{V}(r/N) a_{p+r}^* a_q^* a_p a_{q+r}$$

and $\mathcal{E}_N$ such that for every $\delta > 0$, there exists $C > 0$ such that

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Lower bound: states with small excitation energy have a small expectation for the $\mathcal{N}_+$ operator and therefore they exhibit complete condensation.

Proof of condensation $\quad \mathcal{G}_N = T^* U H_N U^* T$

Consider $\psi_N \in L^2_s(\Lambda^N)$: $\quad \langle \psi_N, H_N \psi_N \rangle \leq 4\pi a_0 N + K$

$$c\mathcal{N}_+ - C \leq \tfrac{1}{2}\mathcal{H}_N - C \leq \mathcal{G}_N - 4\pi a_0 N \leq C(\mathcal{H}_N + 1)$$

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Since

$$N(1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle) = N - \langle \psi_N, a_0^* a_0 \psi_N \rangle \overset{\text{action of } U}{=} \langle \psi_N, U^* \mathcal{N}_+ U \psi_N \rangle$$

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we obtain

$$N\big(1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle\big) \leq C(K+1)$$

Condensation in the ground state

For any $\xi_N \in \mathcal{F}_+^{\leq N}$ we can consider $\psi_N = U^* T \xi_N \in L^2(\mathbb{R}^{3N})$.

We pick $\Omega = \{1, 0, 0, ...\} \in \mathcal{F}_+^{\leq N}$. Hence

$$E_{gs} = \langle \psi_{gs}, H_N \psi_{gs} \rangle \leq \langle U^* T\Omega, H_N U^* T\Omega \rangle$$

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Hence the one-particle reduced density $\gamma_N^{(1)}$ associated with ψ_{gs} is such that

$$1 - \langle \varphi_0, \gamma_N^{(1)} \varphi_0 \rangle \leq \frac{C}{N}$$

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Ground state energy and spectrum for $\beta < 1$

Let $H_N^\beta = \sum_{p\in\Lambda^*} p^2 a_p^* a_p + \frac{\kappa}{2N} \sum_{p,q,r\in\Lambda^*} \widehat{V}(r/N^\beta) a_{p+r}^* a_{q-r}^* a_p a_q$ and consider

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Then $\quad \mathcal{G}_N^\beta = 4\pi a_N^\beta N + \mathcal{H}_N^\beta + \mathcal{E}_N^\beta \quad$ with $\quad \pm\,\mathcal{E}_N^\beta \leq \delta \mathcal{H}_N^\beta + C\kappa(\mathcal{N}_+ + 1)$

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Step 3. Using 2. one can show that for $\beta < 1$ all terms in $\mathcal{G}_N^\beta$ that are not constant or quadratic vanishes on low energy states as $N \to \infty$.

$$\mathcal{G}_N^\beta = C_N^\beta + \mathcal{Q}_N^\beta + \delta_N^\beta \quad \text{with} \quad \pm \delta_N^\beta \leq CN^{-\alpha}(\mathcal{N}_+ + 1)(\mathcal{H}_N^\beta + 1)$$

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Step 4. Diagonalization of the quadratic operator $\mathcal{Q}_N^\beta$

Quadratic Hamiltonian and diagonalization

We have $\mathcal{G}_N^\beta = C_N^\beta + \mathcal{Q}_N^\beta + \delta_N^\beta$ where $\pm\delta_N^\beta \leq CN^{-\alpha}(\mathcal{N}_+ + 1)(\mathcal{H}_N^\beta + 1)$,

$$C_N^\beta = \frac{1}{2}(N-1)\kappa\widehat{V}(0)$$
$$+\sum_{p\in\Lambda_+^*}\Big[p^2\sinh^2\eta_p + \kappa\widehat{V}(p/N^\beta)(\sinh^2\eta_p + \sinh\eta_p\cosh\eta_p) + \tfrac{\kappa}{2N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N^\beta)\eta_p\eta_q\Big]$$

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and $\mathcal{Q}_N^\beta = \sum_{p\in\Lambda_+^*}\big[F_p b_p^* b_p + \frac{1}{2}G_p(\, b_p^* b_{-p}^* + b_p b_{-p}\,)\big]$ with

$$F_p = p^2(\sinh^2\eta_p + \cosh^2\eta_p) + \kappa\widehat{V}(p/N^\beta)(\sinh\eta_p + \cosh\eta_p)^2$$

$$G_p = 2p^2\sinh\eta_p\cosh\eta_p + \kappa\widehat{V}(p/N^\beta)(\sinh\eta_p + \cosh\eta_p)^2 + \frac{k}{N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N^\beta)\eta_q$$

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The operator $\mathcal{Q}_N^\beta$ may be diagonalized using a second generalized Bogoliubov transformation

$$S = \exp\left[\tfrac{1}{2}\sum_{p\in\Lambda_+^*}\tau_p(b_p^* b_{-p}^* - b_p b_{-}p)\right], \; \tanh(2\tau_p) = -\frac{G_p}{F_p}$$

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$$F_p = p^2(\sinh^2\eta_p + \cosh^2\eta_p) + \kappa\widehat{V}(p/N^\beta)(\sinh\eta_p + \cosh\eta_p)^2 \simeq p^2$$
$$G_p = 2p^2\sinh\eta_p\cosh\eta_p + \kappa\widehat{V}(p/N^\beta)(\sinh\eta_p + \cosh\eta_p)^2 + \frac{k}{N}\sum_{q\in\Lambda_+^*}\widehat{V}((p-q)/N^\beta)\eta_q \simeq \frac{1}{p^2}$$

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Diagonal excitation Hamiltonian

Let $\mathcal{M}_N^\beta = S^* \mathcal{G}_N^\beta S : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, then

$$\mathcal{M}_N^\beta = \tilde{E}_N^\beta + \sum_{p \in \Lambda_+^*} \sqrt{|p|^4 + 2|p|^2 \kappa \widehat{V}(0)}\, a_p^* a_p + \rho_{N,\beta}$$

with

$$\tilde{E}_N^\beta = 4\pi(N-1) a_N^\beta + \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[-p^2 - \kappa \widehat{V}(0) + \sqrt{|p|^4 + 2|p|^2 \kappa \widehat{V}(0)} + \frac{\kappa \widehat{V}^2(0)}{2p^2} \right]$$

and

$$\rho_{N,\beta} \leq C N^{-\alpha} (\mathcal{N}_+ + 1)(\mathcal{H}_N^\beta + 1).$$

Diagonal excitation Hamiltonian

Let $\mathcal{M}_N^\beta = S^* \mathcal{G}_N^\beta S : \mathcal{F}_+^{\leq N} \to \mathcal{F}_+^{\leq N}$, then

$$\mathcal{M}_N^\beta = \tilde{E}_N^\beta + \sum_{p \in \Lambda_+^*} \sqrt{|p|^4 + 2|p|^2 \kappa \widehat{V}(0)}\, a_p^* a_p + \rho_{N,\beta}$$

with

$$\tilde{E}_N^\beta = 4\pi(N-1) a_N^\beta + \frac{1}{2} \sum_{p \in \Lambda_+^*} \left[-p^2 - \kappa \widehat{V}(0) + \sqrt{|p|^4 + 2|p|^2 \kappa \widehat{V}(0)} + \frac{\kappa \widehat{V}^2(0)}{2p^2} \right]$$

and

$$\rho_{N,\beta} \leq C N^{-\alpha} (\mathcal{N}_+ + 1)(\mathcal{H}_N^\beta + 1).$$

We compare the eigenvalues of $\mathcal{M}_N^\beta - \tilde{E}_N^\beta$ (*i.e.* the eigenvalues of $H_N^\beta - \tilde{E}_N^\beta$) with those of the quadratic operator

$$\mathcal{D} = \sum_{p \in \Lambda_+^*} \sqrt{|p|^4 + 2|p|^2 \kappa \widehat{V}(0)}\, a_p^* a_p$$

showing that below an energy ζ

$$|\lambda_m - \tilde{\lambda}_m| \leq C N^{-\alpha} (1 + \zeta^3)$$

Extension to the Gross-Pitaevskii regime

For $\beta < 1$, on low energy states, $\mathcal{G}_N^\beta = T^* U H_N^\beta U^* T$ is dominated by its quadratic part:

$$\mathcal{G}_N^\beta = C_N^\beta + \mathcal{Q}_N^\beta + \mathcal{E}_N^\beta \quad \text{with} \quad \pm\, \mathcal{E}_N^\beta \leq C N^{-\alpha} (\mathcal{N}_+ + 1)(\mathcal{H}_N^\beta + 1)$$

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For $\beta = 1$ instead $\mathcal{G}_N^\beta$ contains cubic and quartic contributions of order one.

- **Key fact:** quasi-free states can only approximate the ground state of a dilute Bose gas up to an error of order one [Erdös-Schlein-Yau, '08], [Napiorkòwski-Reuvers-Solovej '15]
- **Challenge:** H_N^{GP} must be conjugated with more complicate maps; in fact an upper bound compatible with the Lee-Huang-Yang prediction has been obtained by using a trial state containing quadratic and cubic correlations [Yau-Yin, '13]

Perspectives

- Proof of condensation in the Gross-Pitaevskii regime without the smallness assumption on $\kappa > 0$
- Extend the results to non-translation-invariant bosonic systems trapped by confining external fields
- Ground state energy and excitation spectrum in the Gross-Pitaevskii regime
- ...
- Validity of Bogoliubov theory for dilute Bose gases in the thermodynamic limit