The shape of the emerging condensate in effective models of condensation

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Effective models of condensation

Particle models:

- Condensation in particle systems: a macroscopic fraction of the particles in a microscopic fraction of state space.
- This means: the value of a suitable 'observable' is the same for a macroscopic fraction of the particles
- Example: BEC, the observable is energy;
- Example: Selection-mutation models, the observable is fitness.
- Proving existence of condensation is very hard.

Effective models:

- Model the dynamics of the relevant quantity directly as a differential or integral equation.
- Condensation in the effective model means that smooth initial conditions converge weakly to measures with a dirac at the relevant place.
- Dynamical condensation is known for a few models.
- We will be interested in the shape of the function (on the right scale) as it approaches a Delta peak.

Kingmans model of selection and mutation

$p_n(\mathrm{d}x)$: fitness distribution of a population. Fitness $x \in [0,1]$.
Effective equation:

$$p_{n+1}(\mathrm{d}x) = (1-\beta)\frac{x}{w_n} p_n(\mathrm{d}x) + \beta r(\mathrm{d}x)$$

with $w_n := \int_0^1 x p_n(\mathrm{d}x)$ (mean fitness)
$0 < \beta < 1$ (mutation rate)
$r(\mathrm{d}x) =$ mutant distribution.

Abstract form:

$$p_{n+1} = \boldsymbol{B}[p_n]p_n + \boldsymbol{C}$$

[Kingman 1978]: if $\gamma = 1 - \beta \int_0^1 \frac{r(\mathrm{d}x)}{1-x} > 0$, then condensation of size γ occurs at $x = 1$, as $t \to \infty$.

[Dereich, Mörters 2013]: If $r(\,(1-h, 1)\,) \sim h^\alpha$, then

$$\lim_{n\to\infty} p_n(\,(1 - x/n, 1)\,) = \frac{\gamma}{\Gamma(\alpha)} \int_0^x y^{\alpha-1}\,\mathrm{e}^{-y}\,\mathrm{d}y.$$

Gamma distribution

A model for Bosons in a bath of Fermions

$F_t(k) =$ density of bosons at energy $k > 0$.
Effective equation:

$$\partial_t F_t(k) = \int_0^\infty b(k,y)\Big(F_t(y)(k^2+F_t(k))\,\mathrm{e}^{-k} - F_t(k)(y^2+F_t(y))e^{-y}\Big)\,\mathrm{d}y$$

with $b > 0$.

Abstract form: $\partial_t F_t(k) = \boldsymbol{B}[F_t](k)F_t(k) + \boldsymbol{C}[F_t](k)$.

[Escobedo, Mischler 99, 01]: If

$$m := \int_0^\infty F_0(k)\,\mathrm{d}k > m_0 := \int_0^\infty \frac{k^2}{\mathrm{e}^k - 1}\,\mathrm{d}k,$$

then convergence of strength $m - m_0$ at $k = 0$ occurs as $t \to \infty$.

[Escobedo, Mischler, Velazquez 03]: For $b = 1$, the scale on which the condensate emerges is $1/t$, and the shape is a Gamma distribution.

Parameter of the Gamma-Distribution may depend on initial condition, explicit representation formula for $b = 1$, formal asymptotic expansions otherwise.

A model for Bosons in a heat bath

$p_t(k) =$ energy distribution of Bosons, $\hat{C} =$ Fourier transform of the heat bath correlation function, $A(z) = \hat{C}(z)(\mathrm{e}^{\beta z} - 1)$. $F(x) = cx^{1/2}$. Effective equation:

$$\partial_t p_t(x) = \int_0^\infty A(y-x) p_t(x) p_t(y)\,\mathrm{d}y$$
$$- \int_0^\infty \hat{C}(y-x) F(y)\, p_t(x)\,\mathrm{d}y + \int_0^\infty \hat{C}(x-y) p_t(y)\, F(x)\,\mathrm{d}y.$$

Abstract form: $\partial_t p_t(x) = \boldsymbol{B}[p_t](x) p_t(x) + \boldsymbol{C}[p_t](x)$.

[Buffet, de Smedt, Pulé 84]: If

$$m := \int_0^\infty p_0(x)\,\mathrm{d}x > m_0 := \int_0^\infty \frac{F(x)}{e^{\beta x} - 1},$$

then the condensation of strength $m - m_0$ occurs at $x = 0$ as $t \to \infty$.

No previous result about the shape of the emerging condensate.

The Boltzmann-Nordheim equation

$f_t(k) =$ energy distribution of weakly interacting bosons.
Effective equation: complicated; but it has the

Abstract form: $\partial_t f_t(k) = \boldsymbol{B}[f_t](k) f_t(k) + \boldsymbol{C}[f_t]$.

[Escobedo, Velazquez 2015]: Equation blows up in finite time.

Nothing about the shape of the condensate is known. This equation is much more singular that the previous ones!

An abstract point of view

Let $(p_t)_{t \geqslant 0}$, $p_t \in L^1([0,\infty))$, solve the equation

$$\partial p_t(x) = \boldsymbol{A}[p_t](x) = \boldsymbol{B}[p_t](x) p_t(x) + \boldsymbol{C}[p_t](x)$$

for $t > 0$ with initial condition $p_0 \in L^1([0,\infty])$.

$$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} : \{\kappa\delta_0 + f \,\mathrm{d}x : \kappa \geqslant 0, f \in L^1(\mathbb{R}^+)\} \to C(\mathbb{R}^+).$$

and we write p_t instead of $0\delta_0 + p_t \mathrm{d}x$.

We say that (p_t) exhibits condensation at $x = 0$ as $t \to \infty$ if

$$\rho_0 := \lim_{\varepsilon \to 0} \liminf_{t \to \infty} \int_0^\varepsilon p_t(x) \,\mathrm{d}x > 0.$$

ρ_0 is then called the mass of the condensate.

We say that convergence to condensation is regular, with bulk $q \in L^1$, if

$$\forall c > 0 : \quad \lim_{t \to \infty} \int_c^\infty |p_t(x) - q(x)| = 0.$$

Main result: assumptions

$$\partial p_t(x) = \boldsymbol{A}[p_t](x) = \boldsymbol{B}[p_t](x)p_t(x) + \boldsymbol{C}[p_t](x)$$

Assumption A1: Assume that
$\boldsymbol{B} : \{\kappa\delta_0 + f\,\mathrm{d}x : \kappa \geqslant 0, f \in L^1(\mathbb{R}^+)\} \to C^1(\mathbb{R}^+)$, and that there is $q \in L^1$, $\alpha > 0$, $c > 0$, $\kappa > 0$ with

$$\boldsymbol{B}[\kappa\delta_0 + q\mathrm{d}x](0) = 0, \quad \partial_x \boldsymbol{B}[\kappa\delta_0 + q\mathrm{d}x](0) < 0,$$
$$\lim_{x\to 0} x^{-\alpha}\boldsymbol{C}[\kappa\delta_0 + q\mathrm{d}x](x) = c.$$

Assumption A2: Assume that for any sequence $(p_n) \subset L^1$ with

$$p_n\,\mathrm{d}x \to \kappa\delta_0 + q\,\mathrm{d}x \quad \text{weakly, and} \quad \lim_{n\to\infty}\int_c^\infty |p_n(x) - q(x)|\,\mathrm{d}x = 0,$$

we have

$$\lim_{n\to\infty} \|\boldsymbol{B}[p_n] - \boldsymbol{B}[q]\|_{C^1([0,\delta])} = 0, \quad \lim_{n\to\infty} \|\boldsymbol{C}[p_n] - \boldsymbol{C}[q]\|_{C([0,\delta])} = 0.$$

Main result: statements

$$\boldsymbol{B}[\kappa\delta_0 + q\mathrm{d}x](0) = 0, \quad \partial_x \boldsymbol{B}[\kappa\delta_0 + q\mathrm{d}x](0) < 0,$$
$$\lim_{x\to 0} x^{-\alpha} \boldsymbol{C}[\kappa\delta_0 + q\mathrm{d}x](x) = c.$$

Assume (p_t) solves $\partial_t p_t = \boldsymbol{A}[p_t]$ and condensates regularly to $\kappa\delta_0 + q\mathrm{d}x$. Assume further that

$$p_0(x) \sim x^{\alpha_0} \quad \text{near } x = 0,$$

with $\alpha_0 > 0$. Then

$$\lim_{t\to\infty} \frac{1}{t} p_t(x/t) = c_1 \, \mathrm{e}^{-\gamma_\infty(0)x} \Big(1_{\{\alpha \leqslant \alpha_0\}} x^\alpha c_2 + 1_{\{\alpha_0 \leqslant \alpha\}} x^{\alpha_0} \eta(0) \Big).$$

The values of c_1, c_2 and $\gamma_\infty(0)$ are known explicitly (see below).

Application to Kingmans model

Kingmans Model (in continuous time):

$$p_{n+1}(\mathrm{d}x) = (1-\beta)\frac{x}{w[p_n]}p_n(\mathrm{d}x) + \beta r(\mathrm{d}x)$$

replaced by

$$\partial_t p_t(\mathrm{d}x) = \Big((1-\beta)\frac{x}{w[p_t]} - 1\Big)p_t(\mathrm{d}x) + \beta r(\mathrm{d}x)$$

with $w[p] = \int_0^1 x p(\mathrm{d}x)$. Stationary solution:

$$q(\mathrm{d}x) = \beta\frac{r(\mathrm{d}x)}{1-x} + \Big(1 - \beta\int_0^1 \frac{r(\mathrm{d}x)}{1-x}\Big)\delta_1(\mathrm{d}x).$$

We have

$$w[q] = 1-\beta, \quad \text{so} \quad \boldsymbol{B}[q](x) = x-1, \quad \boldsymbol{C}[q] = \beta r(\mathrm{d}x).$$

Putting $y = 1-x$ brings this into our form (condensation at zero), and our theorem applies!

Application to the EMV-model

$$\partial_t F_t(k) = \int_0^\infty b(k,y)\Big(F_t(y)(k^2+F_t(k))\,\mathrm{e}^{-k} - F_t(k)(y^2+F_t(y))e^{-y}\Big)\,\mathrm{d}y$$

so

$$\boldsymbol{B}[F](k) = \int_0^\infty b(k,y)\,\mathrm{e}^{-y}\,\Big(F(y)(\,\mathrm{e}^{k-y}\,-1) - y^2\Big)\,\mathrm{d}y,$$

and

$$\boldsymbol{C}[F](k) = k^2\,\mathrm{e}^{-k}\,\int_0^\infty b(k,y)F(y)\,\mathrm{d}y.$$

With $q(k) = \frac{k^2}{\mathrm{e}^k - 1}\,\mathrm{d}k + \kappa\delta_0$ we find

$$\partial_k \boldsymbol{B}[q](0) = -2\int_0^\infty b(0,y)\frac{y^2}{\mathrm{e}^y\,-1}\,\mathrm{d}y - 2\kappa b(0,0) < 0.$$

So the conditions apply, for (A2) it is enough that $b \in C_b^1$.

The BSP model

$\partial_t p_t(x) = \boldsymbol{B}[p_t](x)p_t(x) + \boldsymbol{C}[p_t](x)$ with

$$\boldsymbol{B}[p](x) = \int_0^\infty A(y-x)p(y)\,\mathrm{d}y - \int_0^\infty \hat{C}(y-x)F(y)\,\mathrm{d}y,$$

$$\boldsymbol{C}(x) = F(x)\int_0^\infty \hat{C}(x-y)p(y)\,\mathrm{d}y.$$

With $q(\mathrm{d}x) = \frac{F(x)}{\mathrm{e}^{\beta x}-1} + \kappa\delta_0$ we find

$$\partial_x \boldsymbol{B}[q](0) = -\beta\hat{C}(-y)q(y)\,\mathrm{d}y - \kappa\beta\hat{C}(0) < 0,$$

showing (A1). For (A2), we need $\hat{C} \in C^1$.

Proof part 1: reformulation of assumptions

$$\partial p_t(x) = \boldsymbol{A}[p_t](x) = \boldsymbol{B}[p_t](x) p_t(x) + \boldsymbol{C}[p_t](x)$$

Assume that $p_t \in L^1$ solves this equation, and condenses regularly to $q + \kappa\delta_0$. Then with

$$b_t(x) = \boldsymbol{B}[p_t](x), \quad c_t(x) = \boldsymbol{C}[p_t](x)$$

we have

(B1): $\lim_{t\to\infty} b_t(0) = 0$.

(B2): $\gamma_t(x) := -\frac{1}{x}(b_t(x) - b_t(0))$ (with $x > 0$) is continuous at $x = 0$, and and that there exists a continuos, strictly positive function $\gamma_\infty : [0, \delta] \to \mathbb{R}^+$ such that

$$\lim_{t\to\infty} \sup_{x\in[0,\delta]} |\gamma_t(x) - \gamma_\infty(x)| = 0.$$

(B3): There exists a continuos function c_∞ with $c_\infty(0) > 0$ and

$$\lim_{t\to\infty} \sup_{x\in[0,\delta]} |c_t(x) - c_\infty(x)| = 0.$$

Proof part 1: variation of constant

$$\partial_t p_t = b_t p_t + c_t, \quad p_0(x) = x^{\alpha_0}\eta(x)$$

Variation of constants gives:

$$p_t(x) = \int_0^t \frac{W_t}{W_s} x^\alpha c_s(x)\, \mathrm{e}^{-(t-s)x\bar\gamma_{s,t}(x)}\, \mathrm{d}s + W_t\, \mathrm{e}^{-tx\bar\gamma_{0,t}(x)}\, p_0(x),$$

where

$$W_s = \mathrm{e}^{\int_0^s b_u(0)\,du}, \qquad \bar\gamma_{s,t}(x) = \begin{cases} \frac{1}{t-s}\int_s^t \gamma_r(x)\,\mathrm{d}r & \text{if } t > s \\ \gamma_t(x) & \text{if } t = s. \end{cases}$$

For fixed s, we have $\bar\gamma_{s,t}(x) \to \gamma_\infty(x)$ as $t \to \infty$. Let $\beta = \min\{\alpha, \alpha_0\}$.

Assume that for $Q_t(\beta) := W_t^{-1}(t+1)^{1+\beta}$, $Q_\infty := \lim_{t\to\infty} Q_t(\beta)$ exists and is finite. Then

$$\lim_{t\to\infty} \frac{1}{t} p_t\left(\frac{x}{t}\right) = \frac{\mathrm{e}^{-\gamma_\infty(0)x}}{Q_\infty}\Big(1_{\{\beta=\alpha\}} x^\alpha \int_0^\infty \frac{Q_s(\beta)}{(s+1)^\beta} c_s(0)\,\mathrm{d}s + 1_{\{\beta=\alpha_0\}} x^{\alpha_0}\eta(0)\Big)$$

Proof part 3: convergence of Q_t

$$Q_t = \mathrm{e}^{-\int_0^t b_u(0)\,\mathrm{d}u}\,(t+1)^{1+\beta}.$$

Theorem: if (p_t) exhibits condensation, then Q_t converges.

Proof: Put $\mu_t(\varepsilon) = \int_0^\varepsilon p_t(x)\,\mathrm{d}x$. Then by the solution formula,

$$Q_t(\beta)\mu_\varepsilon(t) = \int_0^t \mathrm{d}s \int_0^\varepsilon \mathrm{d}x\,(t+1)^{1+\beta} x^\alpha c_s(x) \frac{Q_s(\beta)}{(s+1)^{1+\beta}}\,\mathrm{e}^{-(t-s)x\bar\gamma_{s,t}(x)}$$
$$+ \int_0^\varepsilon \mathrm{d}x\,(t+1)^{1+\beta}\,\mathrm{e}^{-tx\bar\gamma_{0,t}(x)}\,x^{\alpha_0}\eta(x) \quad (*)$$

For fixed $K > 0$, with $M_t = \max_{s \leqslant t} Q_s$, we have

$$Q_t\mu_\varepsilon(t) \leqslant C_1(K)M_K + C_2K^{-\beta}M_t + C_3(K)\varepsilon^{1+\alpha}M_t.$$

Picking first K large enough and then $\varepsilon(t)$ so that $\mu_{\varepsilon(t)} > c$ but $\varepsilon(t) \to 0$ we can show that (M_t) is bounded. Using $(*)$ again we can then show convergence.

Conclusions, Observations, Open questions

- We gave a general criterion for 'universal Gamma shape' of the condensate if condensation occurs as $t \to \infty$.
- We can specify when the initial condition dominates the shape, and when the inhomogeniety $\boldsymbol{C}$ does.
- We know no relevant models with condensation at infinity where the criterion fails.
- $p \mapsto \boldsymbol{B}[p]$ was linear (EMV, BSP) or almost trivial (Kingman). What about stronger nonlinearities?
- But our theory does not apply to convergence at finite time (Boltzmann-Nordheim!),
- This cannot be repaired by a rescaling of time, as this leads to a non-autonomous system.
- Obvious question: what is the relevant shape in this case?