Quantum trajectories: invariant measure uniqueness and mixing
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A canonical experiment (S. Haroche’s group)
Definition (Quantum states)

Density matrices:

\[ D := \{ \rho \in \mathbb{M}_d(\mathbb{C}) \mid \rho \geq 0, \text{tr} \rho = 1 \}. \]

Definition (Pure states)

Pure states are the extreme points of \( D \). Namely, \( \rho \in D \) is a pure state iff.
\[ \exists x \in \mathbb{C}^d \setminus \{0\} \text{ s.t.} \]
\[ \rho = P_x := |x\rangle \langle x|. \]

Definition (Metric)

Unitary invariant norm distance:

\[ d(\rho, \sigma) = \| \rho - \sigma \|. \]

Remark

For \( U \in U(d) \), \( d(U\rho U^*, U\sigma U^*) = d(\rho, \sigma) \).
Definition (Completely positive trace preserving (CPTP) maps)

Without conditioning on measurement results the system evolution is given by a CPTP map:

$$\Phi : \mathcal{D} \rightarrow \mathcal{D}$$

$$\rho \mapsto \sum_{j=1}^{\ell} V_j \rho V_j^*$$

with Kraus operators $V_j \in M_d(\mathbb{C})$ for all $j = 1, \ldots, \ell$ s.t. $\sum_{j=1}^{\ell} V_j^* V_j = \text{Id}_d$.

Remark

Seeing $\Phi$ as arising from the interaction of the system with an auxiliary system (probe), Kraus operators $V_j = \langle e_i | U \Psi \rangle := \sum_{j=1}^{\ell} U_{ij} \langle e_j | \Psi \rangle$ with:

- The initial state of the probe $|\Psi\rangle\langle\Psi|$  
- The system–probe interaction $U$  
- The observable measured on the probe $J := \sum_{j=1}^{\ell} j |e_j\rangle\langle e_j|$  

Different observables on the probe give different $V_j$ but same $\Phi$.

$$\Phi(\rho) := \text{tr}_{\text{probe}}(U \rho \otimes P_{\Psi} U^*) = \sum_{j=1}^{\ell} \langle e_j | U \Psi \rangle \rho \langle \Psi | U^* e_j \rangle.$$  

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Indirect measurement

Initial state: $\rho \in \mathcal{D}$

- Evolution unconditioned on the measurement: $\rho \mapsto \Phi(\rho)$. 
Indirect measurement

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- Conditioning on the measurement of $J$:

$$\rho \mapsto \rho' = \frac{V_j \rho V_j^*}{\text{tr}(V_j^* V_j \rho)}, \quad \text{with prob.} \quad \text{tr}(V_j^* V_j \rho)$$
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Remark that:

$$\mathbb{E}(\rho' | \rho) = \Phi(\rho).$$
Repeated interactions

Probes: \( U \)

\( Sys. \)

- Without conditioning on the measurement, after \( n \) interactions: \( \bar{\rho}_n = \Phi^n(\rho) \).
Repeated interactions

- Without conditioning on the measurement, after $n$ interactions: $\tilde{\rho}_n = \Phi^\circ n(\rho)$.
- Given the state after $n-1$ measurements of $J$ is $\rho_{n-1}$, after $n$ measurements of $J$:

$$\rho_n := \frac{V_j \rho_{n-1} V_j^*}{\text{tr}(V_j^* V_j \rho_{n-1})}, \quad \text{with prob. tr}(V_j^* V_j \rho_{n-1}).$$

Equivalently, given $\rho_0 = \rho$, after $n$ measurements of $J$ producing result sequence $j_1, \ldots, j_n$:

$$\rho_n := \frac{V_{j_n} \cdots V_{j_1} \rho_{j_1} V_{j_1}^* \cdots V_{j_n}^*}{\text{tr}(V_{j_1}^* \cdots V_{j_n}^* V_{j_n} \cdots V_{j_1} \rho)}, \quad \text{with prob. tr}(V_{j_1}^* \cdots V_{j_n}^* V_{j_n} \cdots V_{j_1} \rho).$$
Definition (Quantum trajectory)

Given a finite set of $d \times d$ matrices $\{V_j\}_{j=1}^{\ell}$ s.t. $\sum_{j=1}^{\ell} V_j^* V_j = \text{Id}_d$, a quantum trajectory is a realization of the Markov chain of kernel:

$$\Pi(\rho, A) := \sum_{j=1}^{\ell} 1_A \left( \frac{V_j \rho V_j^*}{\text{tr}(V_j^* V_j \rho)} \right) \text{tr}(V_j^* V_j \rho)$$

for any $A \subset \mathcal{D}$ measurable.
Definition (Irreducibility)

The CPTP map $\Phi$ is said irreducible if the only non null orthogonal projector $P$ such that $\Phi(PM_d(\mathbb{C})P) \subset PM_d(\mathbb{C})P$ is $P = \text{Id}_d$.

Theorem (Evans, Høegh-Krohn ’78)

A CPTP map $\Phi : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$ is irreducible iff. $\exists! \rho_{\text{inv.}} \in \mathcal{D}$ s.t. $\rho_{\text{inv.}} > 0$ and $\Phi(\rho_{\text{inv.}}) = \rho_{\text{inv.}}$.

Moreover, if $\Phi$ is irreducible, its modulus 1 eigenvalues are simple and form a finite sub group of $U(1)$. The sub group size $m \in \{1, \ldots, d\}$ is equal to the period of $\Phi$ and $\exists 0 < \lambda < 1$ and $C > 0$ s.t. $\forall \rho \in \mathcal{D}$,

$$\left\| \frac{1}{m} \sum_{r=0}^{m-1} \Phi^{(mn+r)}(\rho) - \rho_{\text{inv.}} \right\| \leq C \lambda^n.$$
Theorem (Kümmerer, Maassen ’04)

Let \((\rho_n)_n\) be a quantum trajectory. Then,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \rho_k = \rho_{\infty} \quad \text{a.s.}
\]

with \(\Phi(\rho_{\infty}) = \rho_{\infty}\). Particularly, if \(\Phi\) is irreducible, \(\rho_{\infty} = \rho_{\text{inv}}\). a.s.
Preliminary results: Purification

For any $\rho \in \mathcal{D}$ let $S(\rho) := -\text{tr}(\rho \log \rho)$ be its von Neumann entropy. Then:

$$S(\rho) = 0 \iff \text{rank}(\rho) = 1 \iff \rho \text{ is a pure state}.$$

**Theorem (Kümmerer, Maassen '06)**

The following statements are equivalent:

1. An orthogonal projector $\pi$ s.t. $\pi V_{j_1}^* \cdots V_{j_p}^* V_{j_p} \cdots V_{j_1}^* \propto \pi$ for all $j_1, \ldots, j_p \in \{1, \ldots, \ell\}$ is of rank 1,
2. For any $\rho_0 \in \mathcal{D}$,

$$\lim_{n \to \infty} S(\rho_n) = 0 \quad \text{a.s.}$$

**Remark**

- If $\pi$ is s.t. $\pi V_{j_1}^* \cdots V_{j_p}^* V_{j_p} \cdots V_{j_1}^* \propto \pi$ for all $j_1, \ldots, j_p \in \{1, \ldots, \ell\}$, there exists unitary matrices $U_{j_1, \ldots, j_p}^\pi$ s.t.

$$V_{j_p} \cdots V_{j_1} \propto U_{j_1, \ldots, j_p}^\pi \pi.$$

- In dimension $d = 2$, either $\lim_{n \to \infty} S(\rho_n) = 0$ a.s., or all the matrices $V_j$ are proportional to unitary matrices.
Uniqueness and convergence towards the invariant measure

Theorem (B., Fraas, Pautrat, Pellegrini '17)

If the following two assumptions are verified,

(Φ-erg.) Φ is irreducible,

(Pur.) Any orthogonal projector π s.t. \( \pi V_{j_1}^* \cdots V_{j_p}^* V_{j_p} \cdots V_{j_1} \pi \propto \pi \) for all \( j_1, \ldots, j_p \in \{1, \ldots, \ell\} \) is of rank 1,

\( \Pi \) accepts a unique invariant probability measure \( \nu_{inv}. \)

Moreover, \( \exists 0 < \lambda < 1 \) and \( C > 0 \) s.t. for any probability measure \( \nu \) over \( D, \)

\[
W_1 \left( \frac{1}{m} \sum_{r=0}^{m-1} \nu \Pi^{mn+r}, \nu_{inv.} \right) \leq C\lambda^n
\]

with \( m \in \{1, \ldots, d\} \) the period of \( \Phi. \)
Products of i.i.d. (Furstenberg, Guivarc’h, Kesten, Le Page, Raugi . . . ’60–’80, Books: Bougerol et Lacroix ’85, Carmona et Lacroix ’90) Markov kernel:

$$\Pi_0(\rho, A) = \sum_{j=1}^{\ell} 1_A \left( \frac{V_j \rho V_j^*}{\text{tr}(V_j^* V_j \rho)} \right) p_j$$

with \((p_j)_{j=1}^{\ell}\) a probability measure over \(\{1, \ldots, \ell\}\).

Generalization (Guivarc’h, Le Page ’01–’16) Markov kernel:

$$\Pi_s(\rho, A) = \mathcal{N}(s)^{-1} \sum_{j=1}^{\ell} 1_A \left( \frac{V_j \rho V_j^*}{\text{tr}(V_j^* V_j \rho)} \right) \left(\text{tr}(V_j^* V_j \rho)\right)^s p_j$$

for \(s \geq 0\) (Q. Traj.: \(s = 1\)).

No assumption that \(\sum_j V_j^* V_j = \text{Id}_d\) but the matrices \(V_j\) need be invertible and a stronger irreducibility condition is assumed.

- \(\{V_j\}_{j=1}^{\ell}\) is strongly irreducible (i.e. no non trivial finite union of proper subspaces is preserved by the matrices \(V_j\)). Then, strong irreducibility \(\implies (\Phi\text{-erg.})\).

- The smallest closed sub semigroup of \(GL_d(\mathbb{C})\) containing \(\{V_j\}_{j=1}^{\ell}\) is contracting (equivalent to \((\text{Pur.})\) for a strongly irreducible family of invertible matrices).
Examples

• Let $p \in ]0, 1[$ and

$V_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ \sqrt{1-p} & 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{pmatrix}, \quad V_4 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}.$

The family $\{V_1, V_2, V_3, V_4\}$ verifies conditions (Φ-erg.) and (Pur.).

• Let,

$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$

The family $\{Z, X\}$ verifies (Φ-erg.) but not (Pur.). There exists uncountably many mutually singular Π-invariant probability measures concentrated on the pure states.

• Let,

$Z = \frac{1}{\sqrt{2}} \begin{pmatrix} e^i & 0 \\ 0 & e^{-i} \end{pmatrix}$ and $X = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos 1 & i \sin 1 \\ i \sin 1 & \cos 1 \end{pmatrix}.$

The family $\{Z, X\}$ verifies (Φ-erg.) but not (Pur.). Nevertheless Π accepts a unique invariant probability measure concentrated on pure states.
Let, \( e_0 = (1, 0)^T \), \( e_1 = (0, 1)^T \) and

\[
V_1 = \begin{pmatrix} 0 & \sqrt{1-p} \\ \sqrt{p} & 0 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} \sqrt{p} & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}
\]

with \( p \in ]0, 1/2[ \). The family \( \{V_1, V_2\} \) defines a CPTP map \( \Phi \) and verifies (\( \Phi\text{-erg.}\)) and (\( \text{Pur.}\)).

**\( \phi \)-irreducibility:** \( \Pi^n(P_{e_0/1}, \{P_{e_0}, P_{e_1}\}) = 1 \) for any \( n \). Hence, if \( \Pi \) is \( \phi \)-irreducible it is so only for \( \phi \ll \frac{1}{2}(\delta_{P_{e_0}} + \delta_{P_{e_1}}) \). Though \( \Pi^n(P_{e+}, \cdot) \) is atomic and \( \Pi^n(P_{e+}, \{P_{e_0}, P_{e_1}\}) = 0 \) for any \( n \) with \( e_+ = \frac{1}{\sqrt{2}}(1, 1)^T \). Hence

\[
\phi(A) > 0 \implies P(\tau_A < \infty | \rho_0 = P_{e+}) = 0
\]

. Particularly,

\[
\left\| \delta_{P_{e_0}} \Pi^n - \delta_{P_{e+}} \Pi^n \right\|_{TV} = 1, \quad \forall n \in \mathbb{N}.
\]

**Contractivity:** For all \( n \in \mathbb{N} \) and \( j_1, \ldots, j_n \in \{1, \ldots, \ell\}, \)

\[
d \left( \frac{V_{j_n} \ldots V_{j_1} P_{e_0} V_{j_n}^* \ldots V_{j_1}^*}{\text{tr}(V_{j_n}^* \ldots V_{j_1}^* V_{j_n} \ldots V_{j_1} P_{e_0})}, \frac{V_{j_n} \ldots V_{j_1} P_{e_1} V_{j_n}^* \ldots V_{j_1}^*}{\text{tr}(V_{j_n}^* \ldots V_{j_1}^* V_{j_n} \ldots V_{j_1} P_{e_1})} \right) = 1.
\]
Proof of uniqueness structure

1. Assuming (**Φ-erg.**), for any \( \Pi \)-invariant probability measure, the distribution of the sequences \((j_n)_n\) of \(J\) measurement results is the same,

2. Assuming (**Pur.**), there exists a process \((\sigma_n)_n\) taking value in \(\mathcal{D}\) and depending only on \((j_n)_n\) s.t.

\[
\lim_{n \to \infty} d(\rho_n, \sigma_n) = 0 \quad a.s.
\]
Lemma

Assume \((\Phi\text{-erg.})\) holds. Then, for any \(\Pi\)-invariant probability measure \(\nu\) over \(D\),
\[
\text{Prob}(j_1, \ldots, j_n|\rho_0 \sim \nu) = \text{tr}(V_{j_1}^* \cdots V_{j_n}^* V_{j_n} \cdots V_{j_1} \rho_{\text{inv}}.)
\]
with \(\rho_{\text{inv.}}\) the unique element of \(D\) s.t. \(\Phi(\rho_{\text{inv.}}) = \rho_{\text{inv.}}\).

Proof.

Given a fixed initial state, the distribution of \(J\) measurement results is given by:
\[
\text{Prob}(j_1, \ldots, j_n|\rho_0 = \rho) = \text{tr}(V_{j_1}^* \cdots V_{j_n}^* V_{j_n} \cdots V_{j_1} \rho).
\]

Linearity in \(\rho\) implies,
\[
\mathbb{E}_\nu [\text{Prob}(j_1, \ldots, j_n|\rho_0 = \rho)] = \text{Prob}(j_1, \ldots, j_n|\rho_0 = \rho_\nu)
\]
with \(\rho_\nu = \mathbb{E}_\nu [\rho]\).

Recall that \(\mathbb{E}_\nu(\rho_1) = \Phi(\rho_\nu)\), but the \(\Pi\)-invariance of \(\nu\) implies \(\mathbb{E}_\nu(\rho_1) = \rho_\nu\). Hence Perron-Frobenius Theorem of positive linear maps imply \(\rho_\nu\) is the unique fixed point state of \(\Phi\).
Set,

\[ W_n := V_{j_n} \ldots V_{j_1}. \]

**Definition**

Let \((M_n)_n\) be the process:

\[ M_n := \frac{W_n^* W_n}{\text{tr}(W_n^* W_n)} \quad \text{if } W_n \neq 0 \]

and arbitrarily fixed in any other case.

**Definition**

Let \(U_n\) and \(D_n\) be two processes s.t. \(U_n D_n = W_n\) is a polar decomposition of \(W_n\).

**Remark**

\[ \rho_n = \frac{W_n \rho W_n^*}{\text{tr}(W_n^* W_n \rho)} = U_n \frac{\sqrt{M_n \rho} \sqrt{M_n}}{\text{tr}(M_n \rho)} U_n^* \quad \text{a.s.} \]
Asymptotic rank one POVM

**Proposition**

Let $\rho_{ch} := \text{Id}_d / d$.

(i) For any probability measure $\nu$ over $\mathcal{D}$,

$$M_\infty := \lim_{n \to \infty} M_n$$

exists a.s. and in $L^1$-norm. Moreover $E(M_\infty | \rho_0 = \rho_{ch}) = \rho_{ch}$.

(ii) The process $M_n$ is a positive bounded martingale w.r.t. $\text{Prob}(\cdot | \rho_0 = \rho_{ch})$. It follows that for any $\rho \in \mathcal{D}$,

$$d\text{Prob}(\cdot | \rho_0 = \rho) = d\text{tr}(M_\infty \rho) d\text{Prob}(\cdot | \rho_0 = \rho_{ch}).$$

(iii) If (Pur.) holds, there exists a random variable $z$ taking value in $\mathbb{C}^k \setminus \{0\}$ s.t.

$$M_\infty = P_z \text{ a.s.}$$

**Remark**

- $z$ depends only on $(j_n)_n$,
- The explicit expression of $d\text{Prob}(\cdot | \rho_0 = \rho) / d\text{Prob}(\cdot | \rho_0 = \rho_{ch})$ implies that,

$$\text{tr}(\rho P_z) > 0 \text{ a.s.}$$
Lemma

Assume (Pur.) holds. Let \((\sigma_n)_n\) be the process taking value in \(\mathcal{D}\) defined by,

\[
\sigma_n = U_n P_z U_n^*.
\]

Then,

\[
\lim_{n \to \infty} d(\rho_n, \sigma_n) = 0 \quad \text{a.s.}
\]

Proof.

\[
\lim_{n \to \infty} U_n^* \rho_n U_n = \lim_{n \to \infty} \frac{\sqrt{M_n \rho} \sqrt{M_n}}{\text{tr}(M_n \rho)} = \frac{P_z \rho P_z}{\text{tr}(P_z \rho)} = P_z \quad \text{a.s.}
\]

The lemma follow from \(\text{tr}(P_z \rho) > 0\) a.s. and

\[
d(\rho_n, \sigma_n) = d(U_n^* \rho_n U_n, P_z).
\]
Uniqueness proof

The uniqueness of the invariant measure follows then from a simple $\epsilon/3$ argument.

Let $\nu_a$ and $\nu_b$ be two $\Pi$-invariant probability measures over $D$.

Since $(\sigma_n)_n$ depends only on the sequence $(j_n)_n$, the first lemma implies:

$$(\sigma_n)_n \text{ w.r.t. } \nu_a \sim (\sigma_n)_n \text{ w.r.t. } \nu_b$$

Then $\rho_n \sim \nu_{a/b}$ and the a.s. convergence $d(\rho_n, \sigma_n) \rightarrow 0$ w.r.t. both $\nu_{a/b}$ implies $\nu_a = \nu_b$. 
Theorem (BFPP '17)

If assumptions (Φ-erg.) and (Pur.) hold, then there exists $0 < \lambda < 1$ and $C > 0$ s.t. for any probability measure $\nu$ over $\mathcal{D}$,

$$W_1 \left( \frac{1}{m} \sum_{r=0}^{m-1} \nu^\Pi^{mn+r}, \nu_{\text{inv.}} \right) \leq C \lambda^n$$

with $m \in \{1, \ldots, d\}$ the period of $\Phi$. 
The proof is again split in two.

- **(Φ-erg.)** $\implies \left\| \frac{1}{m} \sum_{r=0}^{m-1} \Phi^{mn+r}(\rho) - \rho_{inv.} \right\| \leq C\lambda^n \implies$

  $\left\| \frac{1}{m} \sum_{r=0}^{m-1} \text{Prob}(\cdot | \Phi^{mn+r}(\rho_0)) - \text{Prob}(\cdot | \rho_0 = \rho_{inv.}) \right\|_{TV} \leq C\lambda^n.$

- **(Pur.)** $\implies \exists (\hat{\rho}_n)_n$ taking value in $\mathcal{D}$ and depending only on $(j_n)_n$ s.t. for any probability measure $\nu$ over $\mathcal{D},$

  \[ \mathbb{E}_\nu(d(\rho_n, \hat{\rho}_n)) \leq C\lambda^n. \]

The result follows then from an $\epsilon/3$ argument over the expectation of 1-Lipschitz functions and Kantorovich-Rubinstein duality theorem.
Definition

Let \((\hat{P}_n)\) be the sequence of maximum likelihood estimates of the quantum trajectory initial state.

\[
\hat{P}_n := \arg\max_{\rho \in \mathcal{D}} \text{tr}(V_{j_1}^* \ldots V_{j_n}^* V_{j_n} \ldots V_{j_1} \rho)
\]

Proposition

- The estimate \((\hat{P}_n)\) is in general not consistent.
- If assumption (Pur.) holds, then,

\[
\lim_{n \to \infty} \hat{P}_n = P_z \quad \text{a.s.}
\]

Definition

\[
\hat{\rho}_n := \frac{W_n \hat{P}_n W_n^*}{\text{tr}(W_n \hat{P}_n W_n^*)} = U_n \hat{P}_n U_n^*. 
\]

Lemma

Assume (Pur.) holds. Then there exists \(C > 0\) and \(0 < \lambda < 1\) s.t. for any probability measure \(\nu\),

\[
\mathbb{E}_\nu(d(\rho_n, \hat{\rho}_n)) \leq C \lambda^n.
\]
Proof limitations

- The definition of (Pur.) is unsatisfactory. It is difficult to check for explicit matrices $V_j$,
- No information on the continuity of the invariant probability measure.