Decay of correlations in 2d quantum systems

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The results presented are part of a joint work with J. Fröhlich and D.Ueltschi (Ann. Henri Poincaré 18, 2831–2847, (2017)).

Phase transitions and symmetries

- Symmetry: group of transformations that leaves the system unaltered.
- Phase transition due to symmetry breaking: if $T < T_c$ the system favours an ordered state.
- Continuous symmetries (e.g. $\mathbf{U(1)}$) VS Discrete symmetries (e.g. $\mathbb{Z}_2$)

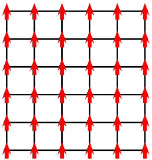

Figure: $T < T_c$

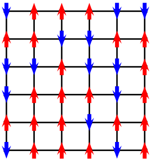

Figure: $T > T_c$

What about dimensions?

Mermin Wagner Theorem: no spontaneous breaking of a **continuous** symmetry can happen at $d \leq 2$ if $T > 0$.

N.B. This statement does not apply to discrete symmetries (e.g. $\mathbb{Z}_2$ symmetry in the ferromagnetic Ising model).

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Phase transitions and correlation functions

- Study of the behaviour of the relevant correlation functions to study the absence or presence of symmetry breaking.
- Expected decay rate in $d = 2$: **power law**.

A general setting

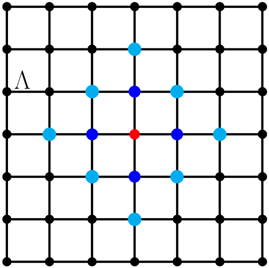
Our aim is to find a general setting for the relevant correlation functions to decay algebraically, with a focus on **quantum models on a 2d lattice**. We will be interested in systems with a **U(1) symmetry**.

We will show how this setting is fulfilled by a great variety of well studied quantum systems.

- N. D. Mermin, H. Wagner, *Absence of ferromagnetism or antiferromagnetism in one- or two- dimensional isotropic Heisenberg models*. Phys. Rev. Lett. 17, 1133- 1136 (1966).
- O. A. McBryan, T. Spencer, *On the decay of correlations in SO(n)-symmetric ferromagnets*. Commun. Math. Phys. 53, 299-302 (1977).
- T. Koma, H. Tasaki, *Decay of Superconducting and Magnetic Correlations in One- and Two-Dimensional Hubbard Models*. Phys. Rev. Lett. 68, 3248 (1992).
- J. Fröhlich, D. Ueltschi, *Some properties of correlations of quantum lattice systems in thermal equilibrium*. Math. Phys. 56, 053302 (2015).

Some notation

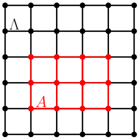
- Lattice: $(\Lambda, \mathcal{E})$ with graph distance.
- $\gamma = \max_{x\in\Lambda} \max_{\ell\in\mathbb{N}} \frac{1}{\ell} |\{y \in \Lambda | d(x,y) = \ell\}$.



- Hilbert space $\mathcal{H}_\Lambda$ - **finite**! (e.g. for quantum spin systems $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda} \mathbb{C}^{2s+1}$).
- Linear operators $\mathcal{B}(\mathcal{H}_\Lambda)$.

Some assumptions

- Local algebras: $\{\mathcal{B}_A\}_{A\subset\Lambda}$.



- Interaction: $\{\Phi_A\}_{A\subset\Lambda}$, $\Phi_A \in \mathcal{B}_A$.
- Hamiltonian: $H_\Lambda = \sum_{A\subset\Lambda} \Phi_A$.
- K-norm of interaction $\{\Phi_A\}_{A\subset\Lambda}$:

$$\|\Phi\|_K = \sup_{y\in\Lambda} \sum_{\substack{A\subset\Lambda \\ \text{s.t.}\, y\in A}} \|\Phi_A\|_\infty (|A|-1)^2 \left(\operatorname{diam}(A)+1\right)^{2+2K(|A|-1)}.$$

- $\langle a \rangle = \frac{\mathrm{Tr}\; a\, e^{-\beta H_\Lambda}}{\mathrm{Tr}\; e^{-\beta H_\Lambda}}$ Gibbs state.

Some assumptions

- U(1) symmetry: $\{S_x\}_{x\in\Lambda}$ such that

$$\left[\Phi_A, \sum_{x\in A} S_x\right] = 0 \quad \forall A \subset \Lambda.$$

- Correlator $O_{xy} \in \mathcal{B}_{\{x,y\}}$ such that $[S_x, O_{xy}] = cO_{xy}$.

Theorem (C.B., J. Fröhlich, D. Ueltschi (2017))

Suppose that the constant γ is finite, and that $\{S_x\}_{x\in\Lambda}$, $(\Phi_A)_{A\subset\Lambda}$, and O_{xy} satisfy the properties above . Then there exist $C > 0$ and $\xi(\beta) > 0$ (uniform with respect to Λ and $x, y \in \Lambda$) such that

$$|\langle O_{xy}\rangle| \leq C\,(d(x,y)+1)^{-\xi(\beta)}\,.$$

Moreover, if there exists a positive constant K such that $\|\Phi\|_K$ is bounded uniformly in Λ, then

$$\lim_{\beta\to\infty} \beta\,\xi(\beta) = \frac{c^2}{8\gamma\|\Phi\|_0}.$$

What does it mean?

For β large enough we have power law decay of correlations with exponent $\propto \frac{1}{\beta}$:

$$|\langle O_{xy}\rangle| \leq \frac{const.}{(d(x,y)+1)^{\frac{const.}{\beta}}}.$$

All the constants are **uniform in** Λ.

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Ex. 1 - SU(2) invariant model

- $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^{2s+1}$.
- $\vec{\mathcal{S}} = (\mathcal{S}^1, \mathcal{S}^2, \mathcal{S}^3)$ spin-s operators, with $\mathcal{S}^i_x = \mathcal{S}^i \otimes \mathbb{1}_{\Lambda\setminus x}$.
- $H_\Lambda = -\sum_{\langle x,y\rangle\in\mathcal{E}}\sum_{k=1}^{2s} c_k(x,y)\left(\vec{\mathcal{S}}_x\cdot\vec{\mathcal{S}}_y\right)^k$.

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Theorem

There exist constants $C > 0$ and $\xi(\beta) > 0$, the latter depending on β, γ, s but not on $x, y \in \Lambda$, such that

$$|\langle \mathcal{S}^j_x \mathcal{S}^j_y \rangle| \leq C\,(d(x,y)+1)^{-\xi(\beta)}\,.$$

The exponent $\xi(\beta)$ is proportional to β^{-1} for β large enough:

$$\lim_{\beta\to\infty} \beta\,\xi(\beta) = (32s\gamma^2)^{-1}.$$

Ex. 2- The Hubbard Model

- $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\text{span}\{\emptyset, \uparrow, \downarrow, \uparrow\downarrow\} \simeq \otimes_{x\in\Lambda}\mathbb{C}^4$.
- Hamiltonian (also long range interaction):

$$H_\Lambda = -\sum_{x,y\in\Lambda}\sum_{\sigma=\uparrow,\downarrow}\frac{t_{xy}}{2}\left(c^\dagger_{\sigma,x}c_{\sigma,y} + c^\dagger_{\sigma,y}c_{\sigma,x}\right) + V(\{n_{\uparrow,x}\}_{x\in\Lambda}, \{n_{\downarrow,x}\}_{x\in\Lambda}).$$

- Two U(1) symmetries generated by $n_x = \sum_{\sigma=\uparrow,\downarrow} n_{\sigma,x}$ and $\Delta_x = n_{\uparrow,x} - n_{\downarrow,x}$.

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Ex. 2 - The Hubbard model

Theorem

Suppose that $t_{xy} = t(d(x,y)+1)^{-\alpha}$ *with* $\alpha > 4$*. Then there exist* $C > 0$, $\xi(\beta) > 0$ *(the latter depending on* β, γ, α, t*, but not on* $x, y \in \Lambda$*) such that*

$$\left.\begin{array}{c} |\langle c^\dagger_{\uparrow,x} c_{\downarrow,x} c^\dagger_{\downarrow,y} c_{\uparrow,y}\rangle| \\ |\langle c^\dagger_{\uparrow,x} c^\dagger_{\downarrow,x} c_{\uparrow,y} c_{\downarrow,y}\rangle| \\ |\langle c^\dagger_{\sigma,x} c_{\sigma,y}\rangle| \end{array}\right\} \leq C(d(x,y)+1)^{-\xi(\beta)}$$

where $\sigma \in \{\uparrow,\downarrow\}$ *in the last line. Furthermore,*

$$\lim_{\beta\to\infty} \beta\,\xi(\beta) = \Big(64\gamma^2|t|\sum_{r\geq 1} r^{-\alpha+3}\Big)^{-1}.$$

- Wide range of applicability!
- Other examples:
  - XXZ model,
  - tJ model,
  - Random loop model,
  - . . .

Ideas of the proof

The proof uses the complex rotation method: "rotate" the correlator and the Hamiltonian with the operator $R = \prod_{z \in \Lambda} e^{\theta_z S_z}$. The "angles" $\{\theta_z\}_{z \in \Lambda}$ are chosen to encode the expected power law decay. We can then estimate $\langle O_{xy} \rangle$ by Trotter's formula and Hölder inequality for matrices.

Conclusions

- Power-law bound for the decay of correlations in a wide class of U(1)-symmetric systems.
- The proof relies on simple ingredients – most of all the complex rotation method.

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Thank you!