Layering in the Ising model

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Alexander, K.S., Dunlop, F., Miracle-Solé, S.:

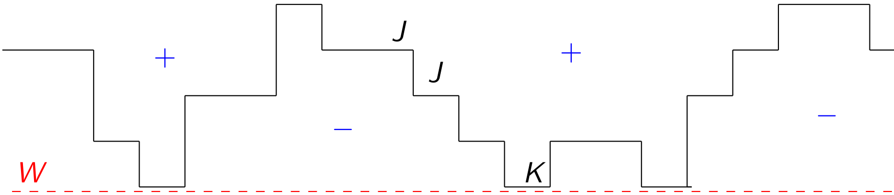
. . .

Layering and wetting transitions for an SOS interface,
arXiv:0908.0321v1 [math-ph]

. . .

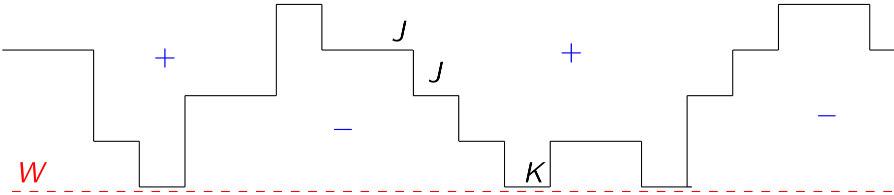
Layering in the Ising model
arXiv:0911.2105v1 [math-ph]

Warwick, November 2009.



$$H_\Lambda(\sigma_\Lambda|\bar\sigma) = -2J|\Lambda_1| + J \sum_{<i,j>\cap\Lambda\neq\emptyset} (1-\sigma_i\sigma_j) + K\sum_{i_3=1}(1+\sigma_i).$$

$$i,j \in \mathbb{Z}^3_+\,, \qquad \Lambda\subset\mathbb{Z}^3_+\,, \qquad \Lambda_1 = \Lambda\cap W\,, \qquad 0<K<J$$



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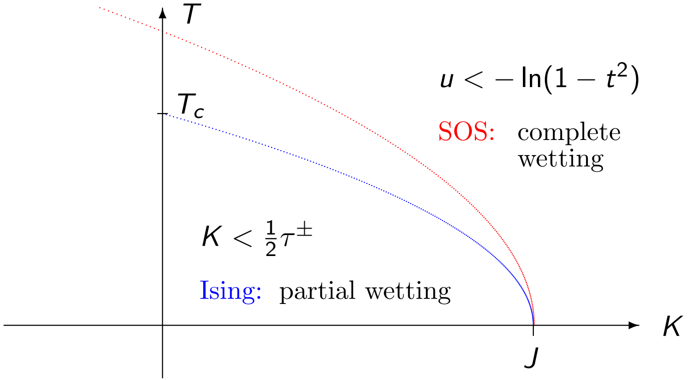
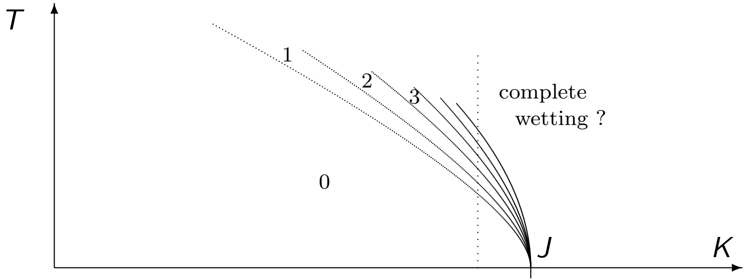
$$i,j\in\mathbb{Z}^3_+\,, \qquad \Lambda\subset\mathbb{Z}^3_+\,, \qquad \Lambda_1=\Lambda\cap W\,, \qquad 0<K<J$$

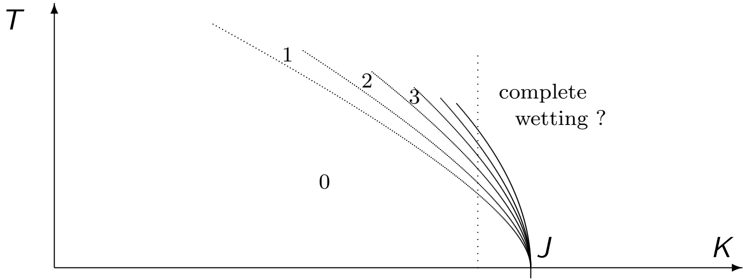
$$t=e^{-4\beta J}, \qquad u=2\beta(J-K), \qquad \text{bubble}=t^3, \qquad \text{contact}=e^u$$



- Fröhlich - Pfister '87
- Chalker '82



Phase $n \simeq n$ layers of $-$spins



Phase $n \simeq n$ layers of $-$spins

- Ising: formal power series in $t = e^{-4\beta J}$, indicating a range of stability of phase n at low T, for $n = 0, 1, 2, 3, 4, 5, 6, 7$
- SOS: convergent 2-scale cluster expansion, proving a range of stability of phase n, for each $n \geq 0$

Boundary condition $\bar{\sigma} = n$, with $n = 0, 1, 2, \ldots$, defined by

$$\bar{\sigma}_i = -1 \text{ if } i_3 \leq n, \qquad \bar{\sigma}_i = +1 \text{ if } i_3 > n,$$

Associated restricted ensemble partition function:

$$Z_n^\Lambda = \sum_{\sigma_\Lambda}{}' e^{-\beta H_\Lambda(\sigma_\Lambda|\bar{\sigma})},$$

Surface free energy density $f_n \implies$

$$f_n - f_{n+1} = \lim_{\Lambda \nearrow Z_+^3} -\frac{1}{|\Lambda_1|} \log \frac{Z_n^\Lambda}{Z_{n+1}^\Lambda}$$

Bubbles and interface excitations = contours = polymers = γ.

γ boundary of a maximal connected set of points where the spin differs from its ground state value.

$$Z_n^\Lambda = \sum_{\{\gamma\}} \prod_\gamma \varphi(\gamma)\,, \qquad \varphi(\gamma) = t^{\frac{1}{2}|\gamma| - |\gamma\cap I_n|} e^{u|\gamma\cap W|}\,, \qquad n \geq 1$$

$$\log(Z_n^\Lambda) = \sum_\omega \varphi^T(\omega)\,, \qquad \varphi^T(\omega) = \prod_{\gamma\in\omega}\left(\frac{1}{n_\gamma!}\varphi(\gamma)^{n_\gamma}\right)\sum_G (-1)^l$$

$$\log(Z_n^\Lambda) = \sum_{\omega \in I_n, W} \varphi^T(\omega) + \sum_{\substack{\omega \in I_n, \\ \omega \approx W}} \varphi_0^T(\omega) + \sum_{\substack{\omega \in W, \\ \omega \sim I_n}} \varphi_1^T(\omega) + \sum_{\substack{\omega \approx W \\ \omega \sim I_n}} \varphi_2^T(\omega)$$

= Interface touching Wall + Interface not touching Wall + Bubble touching Wall + Bubble not touching Wall (clusters !)

$$\varphi_0(\gamma) = t^{\frac{1}{2}|\gamma| - |\gamma \cap I_n|}, \qquad \varphi_1(\gamma) = t^{\frac{1}{2}|\gamma|} e^{u|\gamma \cap \{z = \frac{1}{2}\}|}, \qquad \varphi_2(\gamma) = t^{\frac{1}{2}|\gamma|}$$

$$\log(Z_n^\Lambda) = \sum_{\omega\in I_n, W} \varphi^T(\omega) + \sum_{\substack{\omega\in I_n,\\ \omega\approx W}} \varphi_0^T(\omega) + \sum_{\substack{\omega\in W,\\ \omega\sim I_n}} \varphi_1^T(\omega) + \sum_{\substack{\omega\approx W\\ \omega\sim I_n}} \varphi_2^T(\omega)$$

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$$\varphi_0(\gamma) = t^{\frac{1}{2}|\gamma| - |\gamma\cap I_n|}, \qquad \varphi_1(\gamma) = t^{\frac{1}{2}|\gamma|} e^{u|\gamma\cap\{z=\frac{1}{2}\}|}, \qquad \varphi_2(\gamma) = t^{\frac{1}{2}|\gamma|}$$

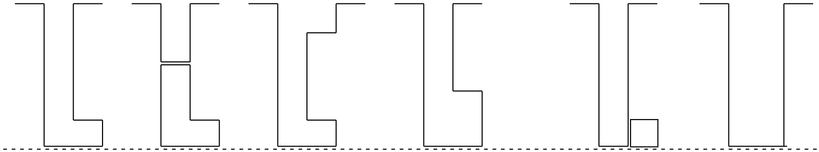
Inclusion/Exclusion $\Longrightarrow$

$$\log(Z_n^{\Lambda_1^\infty}/Z_{n+1}^{\Lambda_1^\infty}) = \sum_{\omega\in I_n, W} \varphi^T(\omega) - \sum_{\substack{\omega\in I_n,\\ \omega\not\approx W_0\\ \omega\approx W_{-1}}} \varphi_0^T(\omega) - \sum_{\omega\in I_{n+1}, W} \varphi^T(\omega)$$

$$-\Bigg(\sum_{\substack{\omega\in W,\\ \omega\not\sim I_n\\ \omega\sim I_{n+1}}} \varphi_1^T(\omega) - \sum_{\substack{\omega\in W,\\ \omega\not\sim I_{n+1}\\ \omega\sim I_n}} \varphi_1^T(\omega)\Bigg) + \Bigg(\sum_{\substack{\omega\not\approx W,\\ \omega\not\sim I_n}} \varphi_2^T(\omega) - \sum_{\substack{\omega\not\approx W,\\ \omega\not\sim I_{n+1}}} \varphi_2^T(\omega)\Bigg)$$

$$t^{-2n}(f_{n+1} - f_n) = 1 + \dots$$

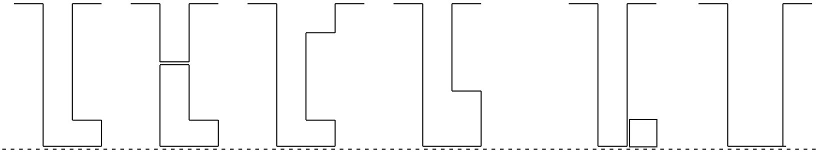
$+$



$$+4t^2 \quad -4(n-1)t^3 \quad +16t^3 \quad +4t^3 \quad -5t^3 \quad +2t^n$$

$$t^{-2n}(f_{n+1} - f_n) = 1 + \dots$$

$+$



$+4t^2 \quad -4(n-1)t^3 \quad +16t^3 \quad +4t^3 \quad -5t^3 \quad +2t^n$

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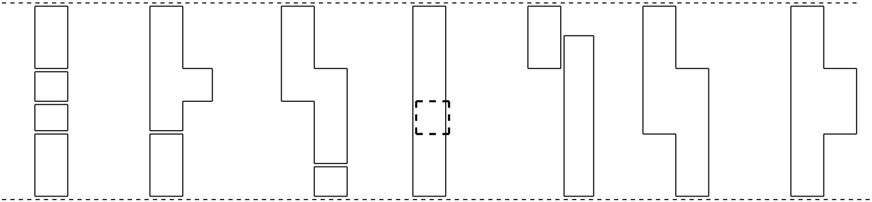
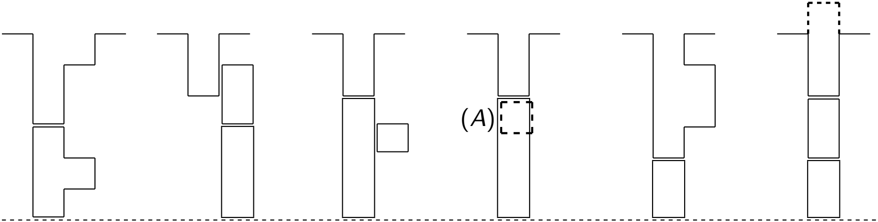
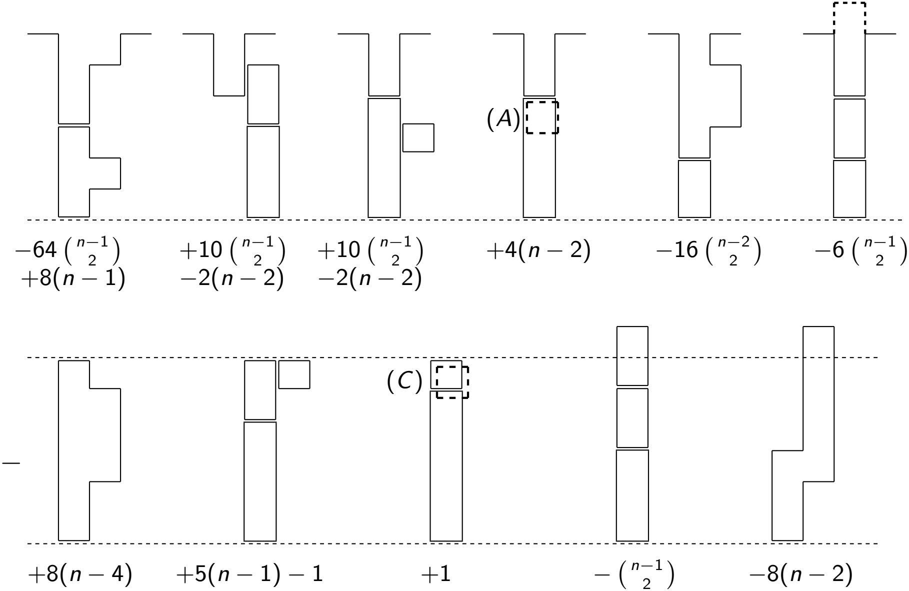
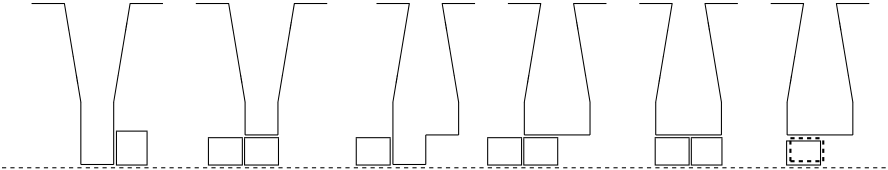


$-\binom{n-1}{3} t^4 \; -4(n-1)^2 t^4 \; -8\binom{n-1}{2} t^4 \; -5(n-1)t^4 \; -5(n-2)t^4 \; +4(n-3)t^4 \; +4(n-2)t^4$

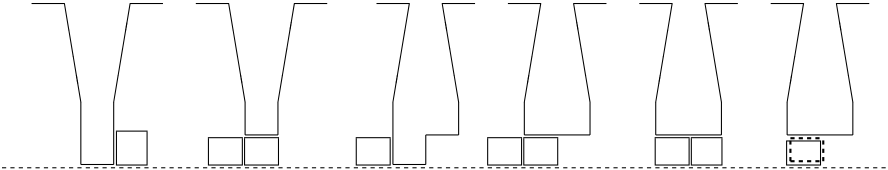


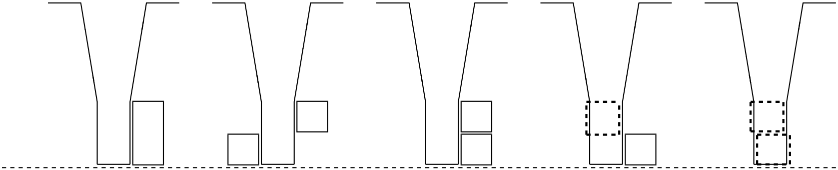
$-64\binom{n-1}{2} + 8(n-1)$ $\quad +10\binom{n-1}{2} - 2(n-2)$ $\quad +10\binom{n-1}{2} - 2(n-2)$ $\quad +4(n-2)$ $\quad -16\binom{n-2}{2}$ $\quad -6\binom{n-1}{2}$


(A)
$-64\binom{n-1}{2}$ $+8(n-1)$
$+10\binom{n-1}{2}$ $-2(n-2)$
$+10\binom{n-1}{2}$ $-2(n-2)$
$+4(n-2)$
$-16\binom{n-2}{2}$
$-6\binom{n-1}{2}$
(C)
$-$
$+8(n-4)$
$+5(n-1)-1$
$+1$
$-\binom{n-1}{2}$
$-8(n-2)$



$$Q^2_{n+1} = \quad -5t^3 P_n \quad +5t^4 P_n \quad -10t^3 Q^1_n \quad +6t^4 Q^1_n \quad +2t^4 Q^1_n \quad +2t^4 Q^1_n$$





$$Q_{n+1}^2 = -5t^3 P_n + 5t^4 P_n - 10t^3 Q_n^1 + 6t^4 Q_n^1 + 2t^4 Q_n^1 + 2t^4 Q_n^1 + t^2 Q_n^2 - 8t^3 Q_n^{2a} - 4t^3 Q_n^{2a} - 8t^3 Q_n^{2b} - 4t^3 Q_n^{2b} + \ldots$$

Let $t = e^{-4\beta J} \ll 1$ and $u = 2\beta(J - K) = O(t^2)$. We find the following approximation to the coexistence (first order transition) lines starting from $(t = 0, u = 0)$:

$$0/1: \quad u = -\ln(1 - t^2) + t^3 + O(t^4)$$

$$1/2: \quad u = -\ln(1 - t^2) - t^3 + 5t^4 + O(t^5)$$

$$2/3: \quad u = -\ln(1 - t^2) - t^3 + 4t^4 - 4t^5 + O(t^6)$$

$$3/4: \quad u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{51}{2}t^6 + O(t^7)$$

$$4/5: \quad u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{47}{2}t^6 - 51t^7 + O(t^8)$$

$$5/6: \quad u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{47}{2}t^6 - 53t^7 + 144t^8 + O(t^9$$

$$6/7: \quad u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{47}{2}t^6 - 53t^7 + 142t^8 + (B_9 - 2)t^9 + O(t^{10})$$

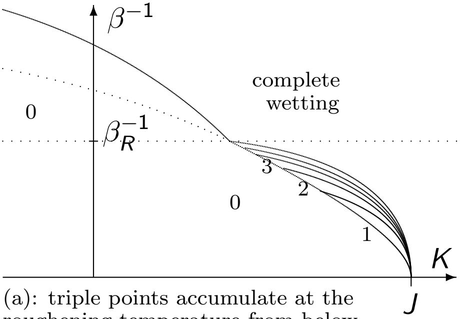
$$7/8: \quad u = -\ln(1 - t^2) - t^3 + 4t^4 - 6t^5 + \frac{47}{2}t^6 - 53t^7 + 142t^8 + B_9 t^9 + O(t^{10})$$

Theorem

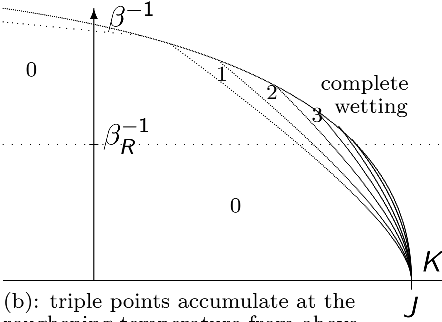
$\forall n \geq 0, \forall \varepsilon > 0, \; \exists t_0(n, \epsilon) > 0$ such that, if $0 < t < t_0(n, \epsilon)$ and

$$\begin{aligned} -\ln(1-t^2) + (2+\epsilon)t^{n+3} < u < -\ln(1-t^2) + (2-\epsilon)t^{n+2}, \qquad & n \geq 1, \\ -\ln(1-t^2) + (2+\epsilon)t^3 < u < \sqrt{t}, \qquad & n = 0, \end{aligned}$$

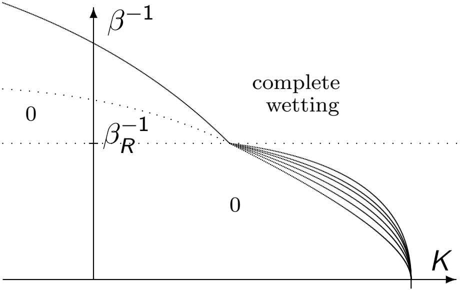
then there is a unique translation invariant Gibbs state μ_n, a pure phase associated to the level n, that satisfies $\mu_n(\{\phi_x \neq n\}) = O(t^2)$ for any given x.



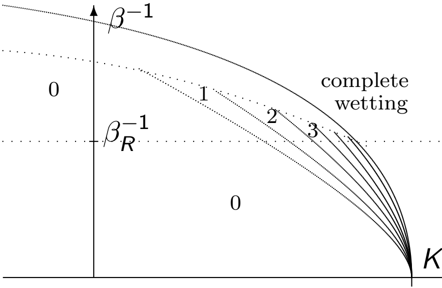
(a): triple points accumulate at the roughening temperature from below



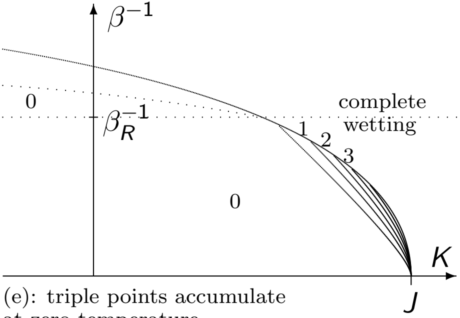
(b): triple points accumulate at the roughening temperature from above



(d): multiphase point at the roughening temperature



(c): critical points accumulate at the roughening temperature from above



(e): triple points accumulate
at zero temperature

Lemma (SOS): let $k \geq 8$ and define the k-restricted ensemble by:

diameter of contours $\leq 3k+3\}$

Let $t \leq (3k+3)^{-4}$ and $u \leq t^{1/2}$. Then the expansion of the restricted free energy is an absolutely convergent power series in t. Choosing $s = te^{2t^{1/4}}$ and $\mu(\gamma) = \varphi_{s,0}(\gamma)$, we have

$$|\varphi(\gamma)| \leq \mu(\gamma) \exp\Big(-\sum_{\gamma' : \gamma' \not\sim \gamma} \mu(\gamma')\Big),$$

$$\sum_{X : \gamma \in X} |\varphi_u^{\mathrm{T}}(X)| \leq \mu(\gamma),$$

$$\sum_{X : \gamma \in X} X(\gamma) |\varphi_u^{\mathrm{T}}(X)| \leq \varphi(\gamma) e^{\sum_{\gamma' : \gamma' \not\sim \gamma} \mu(\gamma')} \leq e^{\mu(\gamma)} - 1.$$

Convergence of 2-scale cluster expansion requires

$$(16t)^{3k+4} < \frac{\varepsilon}{4} t^{3n+3},$$

which is satisfied as soon as $k \geq \max(8, 2n)$ and $t \leq \varepsilon^{1/12}/2000$.

And we still need, from the restricted ensemble expansion,

$$t \leq (3k+3)^{-4}$$

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