Min- and Max-Relative Entropies

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- **Focus :** on an important quantity which arises in
  Quantum Mechanics & Quantum Information Theory

  Quantum Entropy or **von Neumann Entropy**

- and its parent quantity:

  Quantum Relative Entropy

- & on new quantum relative entropies :

  Min- & Max-Relative Entropies

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Quantum Entropy or von Neumann Entropy

- In 1927 von Neumann introduced the notion of a mixed state, represented by a density matrix: $\rho \geq 0; \mathrm{Tr}\,\rho = 1$

$$\rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i|; \qquad \rho \leftrightarrow \{p_i, |\psi_i\rangle\}$$

- & defined its entropy: $$S(\rho) := -\mathrm{Tr}\,(\rho \log \rho)$$

- Why? - To extend the classical theory of Statistical Mechanics developed by Gibbs, Boltzmann et al to the quantum domain;
- not for the development of **Quantum Information Theory**

- In fact, this was well before Shannon laid the foundations of **Classical Information Theory** (1948).

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$$S(\rho) := -\mathrm{Tr}\,(\rho \log \rho)$$

$$\rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i|; \qquad \langle\psi_i|\psi_j\rangle \neq \delta_{ij}$$

$S(\rho) = 0$ if and only if ρ is a pure state:

$$\rho = |\Psi\rangle\langle\Psi|$$

$\therefore S(\rho) =$ a measure of the "mixedness" of the state ρ

Relation with Statistical Mechanics

- The finite volume Gibbs state given by the density matrix:

$$\rho_\Lambda^{Gibbs} = \frac{e^{-\beta H_\Lambda}}{\mathrm{Tr}\, e^{-\beta H_\Lambda}}$$

maximizes the von Neumann entropy, given the expected value of the energy

i.e., the functional : $\rho \mapsto S(\rho_\Lambda) - \beta \mathrm{Tr}\,(\rho_\Lambda H_\Lambda)$

is maximized if and only if $\rho_\Lambda = \rho_\Lambda^{Gibbs}$

$$\log Z_\Lambda = \max_{\rho_\Lambda} \left(S(\rho_\Lambda) - \beta \mathrm{Tr}\,(\rho_\Lambda H_\Lambda) \right)$$

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- In 1948 Shannon defined the entropy of a random variable

- Let $X \sim p(x); \quad x \in J; \qquad J =$ a finite alphabet

- Shannon entropy of X; $$H(X) := -\sum_{x \in J} p(x) \log p(x)$$

$$H(X) \equiv H\big(\{p(x)\}\big)$$

- Supposedly von Neumann asked Shannon to call this quantity entropy, saying

"You should call it 'entropy' for two reasons; first, the function is already in use in thermodynamics; second, and most importantly, most people don't know what entropy really is, & if you use 'entropy' in an argument, you will win every time."

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Relation between Shannon entropy & thermodynamic entropy

$$S := k \log \Omega$$

total # of microstates

- Suppose the r^{th} microstate occurs with prob. p_r

.... consider ν replicas of the system

- Then on average $\nu_r \approx [\nu p_r]$ replicas are in the r^{th} state

- **Total # of microstates** $\Omega = \dfrac{\nu!}{\nu_1! \nu_2! .. \nu_r!} \simeq \dfrac{\nu^{\nu}}{\nu_1^{\nu_1} \nu_2^{\nu_2} .. \nu_r^{\nu_r}}$ (by Stirling's)

- Th.dyn.entropy of compound system of ν replicas

$$S_\nu = -k\nu \sum_r p_r \log p_r$$

$$S = S_\nu / \nu = -k \sum_r p_r \log p_r = k\, H(\{p_r\})$$

Shannon entropy

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Relation between Shannon entropy & von Neumann entropy

$$S(\rho) := -\mathrm{Tr}\,(\rho \log \rho); \qquad \rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i|;$$

$$\rho \leftrightarrow \{p_i, |\psi_i\rangle\}; \qquad \langle\psi_i|\psi_j\rangle \neq \delta_{ij}$$

eigenvalues

- **BUT** $\rho = \rho^\dagger$, Spectral decomposition $\rho = \sum_{i=1}^{n} \lambda_i \Pi_i;$

$\rho \geq 0;$ $\mathrm{Tr}\,\rho = 1$ $\Rightarrow$ $\lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1;$ $\{\lambda_i\}_{i=1}^{n} =$ a probability distribution

$$S(\rho) := -\sum_{i=1}^{n} \lambda_i \log \lambda_i = H(\{\lambda_i\})$$

- *Classical Information Theory* [Shannon 1948]

  = Theory of information-processing tasks

  e.g. storage & transmission of information

- *Quantum Information Theory*:

  A study of how such tasks can be accomplished using quantum-mechanical systems

- *not just a generalization of Classical Information Theory to the quantum realm!*

The underlying quantum mechanics → *novel features!* *which have no classical analogue!*

- In Quantum Information Theory, information is carried by *physical states of quantum-mechanical systems*:

e.g. polarization states of photons, spin states of electrons etc.

Fundamental Units

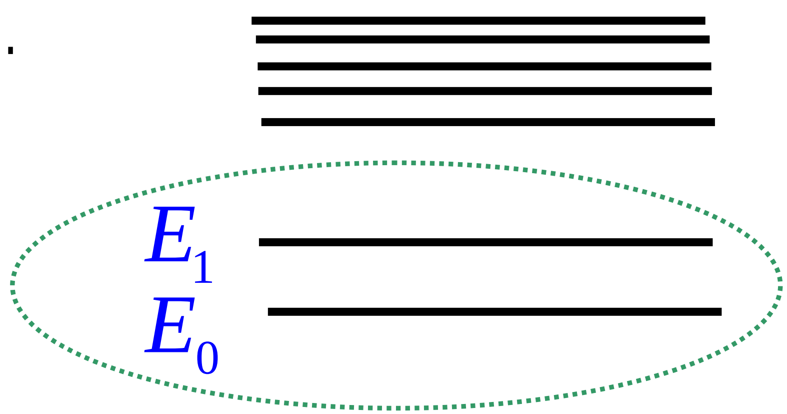
- Classical Information Theory: a bit ; takes values 0 and 1
- Quantum Information Theory: *a qubit*

  = *state of a two-level quantum-mechanical system*

Physical representation of a qubit

- Any *two-level system*; e.g. spin states of an electron, polarization states of a photon......

A multi-level system which has 2 states which can be effectively decoupled from the rest;



Operational Significance of the Shannon Entropy

= *optimal rate of data compression for a classical i.i.d. (memoryless) information source*

successive signals emitted by source : indept. of each other

Modelled by a sequence of i.i.d. random variables

$$U_1, U_2, \ldots, U_n \qquad U_i \sim p(u) \qquad u \in J$$

- signals emitted by the source = $(u_1, u_2, \ldots, u_n)$

Shannon entropy of the source:

$$H(U) := -\sum_{u \in J} p(u) \log p(u)$$

Operational Significance of the Shannon Entropy

- *(Q) What is the optimal rate of data compression for such a source?*

[min. # of bits needed to store the signals emitted per use of the source] *(for reliable data compression)*

- *Optimal rate is evaluated in the asymptotic limit* $n \to \infty$

$n =$ *number of uses of the source*

- *One requires* $p_{error}^{(n)} \to 0 \; ; \; n \to \infty$

- *(A) optimal rate of data compression =* $H(U)$

Shannon entropy of the source

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Operational Significance of the von Neumann Entropy

> *= optimal rate of data compression for a memoryless (i.i.d.) quantum information source*

- A quantum info source emits:

signals (pure states) $|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_k\rangle \in \mathcal{H}$

with probabilities $p_1, p_2, \ldots, p_k$

Hilbert space

- Then source characterized by: $\{\rho, \mathcal{H}\}$

density matrix

$$\rho = \sum_{i=1}^{k} p_i |\psi_i\rangle\langle\psi_i|$$

$$\langle\psi_i|\psi_j\rangle \neq \delta_{ij}$$

To evaluate data compression limit :

Consider a sequence $\{\rho_n, \mathcal{H}_n\}$

If the quantum info source is memoryless (i.i.d.)

$$\mathcal{H}_n = \mathcal{H}^{\otimes n}; \qquad \boxed{\rho_n = \rho^{\otimes n}} \qquad \rho \in \mathcal{H}$$

Optimal rate of data compression $= S(\rho)$

- *NOTE: Evaluated in the asymptotic limit* $n \to \infty$

  $n =$ *number of uses of the source*

  One requires $p_{err}^{(n)} \underset{n\to\infty}{\longrightarrow} 0$

e.g. A memoryless quantum info source emitting qubits

- *Characterized by*

$$\{\rho_n, \mathcal{H}_n\}; \quad \boxed{\rho_n = \rho^{\otimes n}}; \quad \mathcal{H}_n = \mathcal{H}^{\otimes n};$$

- Or simply by $\{\rho, \mathcal{H}\}$

- Consider n successive uses of the source ; n qubits emitted
- Stored in m_n qubits (*data compression*)

$$\boxed{\textit{rate of data compression} = \frac{m_n}{n}}$$

Optimal rate of data compression $R_\infty := \lim_{n\to\infty} \frac{m_n}{n}$ $= S(\rho)$

under the requirement that $p_{error}^{(n)} \underset{n\to\infty}{\longrightarrow} 0$

Quantum Relative Entropy

- A fundamental quantity in Quantum Mechanics & Quantum Information Theory is the Quantum Relative Entropy

  of ρ w.r.t. σ, $\rho \geq 0, \ \mathrm{Tr}\, \rho = 0, \ \sigma \geq 0$:

$$\boxed{S(\rho \,\|\, \sigma) := \mathrm{Tr}\, \rho \log \rho - \mathrm{Tr}\, \rho \log \sigma}$$

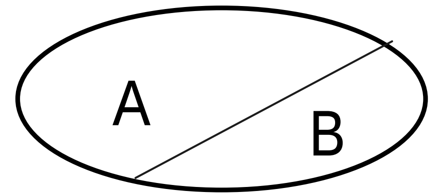
  well-defined if $\mathrm{supp}\, \rho \subseteq \mathrm{supp}\, \sigma$

- It acts as a parent quantity for the *von Neumann entropy*:

$$\boxed{S(\rho) := -\mathrm{Tr}\, \rho \log \rho = -S(\rho \,\|\, I)} \qquad (\sigma = I)$$

- It also acts as a parent quantity for other entropies:

  e.g. for a bipartite state ρ_{AB} :



  - *Conditional entropy*

$$S(A|B) := S(\rho_{AB}) - S(\rho_B) = -S(\rho_{AB} \| I_A \otimes \rho_B)$$

  - *Mutual information*

$$\rho_B = \mathrm{Tr}_A\, \rho_{AB}$$

$$I(A:B) := S(\rho_A) + S(\rho_B) - S(\rho_{AB}) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$$

Some Properties of $S(\rho \| \sigma)$

- Klein's inequality:

  "distance"

$$S(\rho \| \sigma) \geq 0$$
$$= 0 \text{ if \& only if } \rho = \sigma$$

- Joint convexity:

$$S\Big(\sum_k p_k \rho_k \Big\| \sum_k p_k \sigma_k\Big) \leq \sum_k p_k S(\rho_k \| \sigma_k)$$

- Monotonicity under completely positive trace preserving (CPTP) maps Λ:

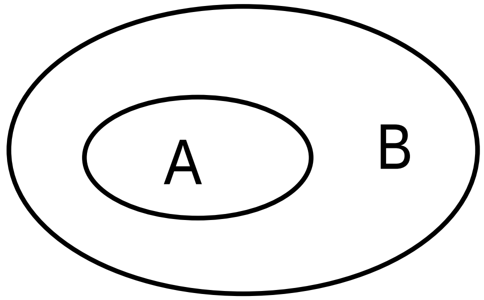
$$S(\Lambda(\rho) \| \Lambda(\sigma)) \leq S(\rho \| \sigma)$$

  - *What is a CPTP map?*

For an isolated quantum system: time evolution is unitary

-- dynamics governed by the Schroedinger equation.

- In Quantum Info. Theory one deals with open systems
  - unavoidable interactions between system (A) & its environment (B)
  - Time evolution not unitary in general



Most general description of time evolution of an open system

given by a completely positive trace-preserving (CPTP) map Λ

$$\Lambda : \rho \to \sigma$$

density operators

describes discrete state changes resulting from any allowed physical process : *a superoperator*

Properties satisfied by a superoperator $\Lambda : \rho \to \sigma$

- Linearity: $\Lambda(\sum_k p_k \rho_k) = \sum_k p_k \Lambda(\rho_k)$

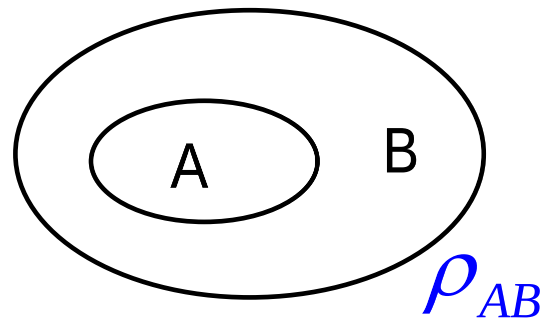
- Positivity: $\sigma = \Lambda(\rho) \geq 0$

- Trace-preserving: (TP) $\mathrm{Tr}\,\sigma = \mathrm{Tr}\,\Lambda(\rho) = \mathrm{Tr}\,\rho = 1$

- Complete positivity: $\mathcal{H}_{\mathrm{A}} \otimes \mathcal{H}_{\mathrm{B}}$

  (CP)

  $$(\Lambda_A \otimes \mathrm{id}_B)(\rho_{AB}) \geq 0$$



- Kraus Representation Theorem: $\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger;$ $\sum_k A_k^\dagger A_k = I$

Monotonicity of Quantum Relative Entropy under CPTP map Λ :

v.powerful!

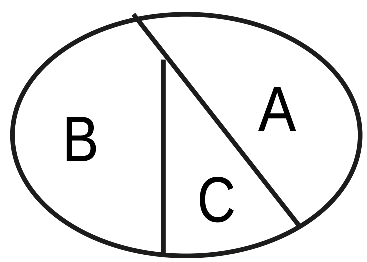
$$S(\Lambda(\rho) \| \Lambda(\sigma)) \leq S(\rho \| \sigma)$$ ....(1)

- Many properties of other entropic quantities can be proved using (1)

  - e.g. *Strong subadditivity of the von Neumann entropy*

- Conjecture by Lanford, Robinson – proved by Lieb & Ruskai '73

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$$



Outline of the rest

- *Define 2 new relative entropy quantities*
- *Discuss their properties and operational significance*
- *Give a motivation for defining them*
- *Define 2 entanglement monotones (I will explain what that means)*
- *Discuss their operational significance*

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Two new relative entropies

- *Definition 1 :* The max- relative entropy of a density matrix ρ & a positive operator σ is

$$S_{\max}(\rho\,||\,\sigma) := \log\left(\min\{\lambda : \rho \leq \lambda\sigma\}\right)$$

$$(\lambda\sigma - \rho) \geq 0$$

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- *Definition 2:* The min- relative entropy of a state ρ & a positive operator σ is

$$S_{\min}(\rho \| \sigma) := -\log \operatorname{Tr}(\pi_\rho \sigma)$$

where π_ρ denotes the projector onto the support of ρ $(\operatorname{supp} \rho)$

- *Remark:* The min- relative entropy

$$S_{\min}(\rho \| \sigma) := -\log \operatorname{Tr} \pi_\rho \sigma$$

(where π_ρ denotes the projector onto $\operatorname{supp} \rho$)

is expressible in terms of: *quantum relative Renyi entropy*

of order α with $\alpha \neq 1$

$$S_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log \operatorname{Tr} \rho^\alpha \sigma^{1-\alpha}$$

as follows:

$$S_{\min}(\rho \| \sigma) = \lim_{\alpha \to 0^+} S_\alpha(\rho \| \sigma)$$

Min- and Max-Relative Entropies

- Like $S(\rho\,||\,\sigma)$ we have

$$S_*(\rho\,||\,\sigma) \geq 0$$ for $* = \max,\ \min$

$$S_*(\Lambda(\rho)\,||\,\Lambda(\sigma)) \leq S_*(\rho\,||\,\sigma)$$ for any CPTP map Λ

- Also $$S_*(\rho\,||\,\sigma) = S_*(U\rho U^\dagger\,||\,U\sigma U^\dagger)$$ for any unitary operator U

- Most interestingly

$$S_{\min}(\rho\,||\,\sigma) \leq S(\rho\,||\,\sigma) \leq S_{\max}(\rho\,||\,\sigma)$$

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- The min-relative entropy is jointly convex in its arguments:

For two mixtures of states $\rho = \sum_{i=1}^{n} p_i \rho_i$ & $\sigma = \sum_{i=1}^{n} p_i \sigma_i$

$$S_{\min}(\rho \| \sigma) \leq \sum_{i=1}^{n} p_i S_{\min}(\rho_i \| \sigma_i)$$

as for $S(\rho \| \sigma)$

- The max-relative entropy is quasiconvex:

$$S_{\max}(\rho \| \sigma) \leq \max_{1 \leq i \leq n} S_{\max}(\rho_i \| \sigma_i)$$

Min- and Max- entropies

$$H_{\min}(\rho) := -S_{\max}(\rho \| I) = -\log \|\rho\|_\infty$$

$$H_{\max}(\rho) := -S_{\min}(\rho \| I) = \log \operatorname{rank}(\rho)$$

analogous to: $S(\rho) = -S(\rho \| I)$

- For a bipartite state ρ_{AB} :

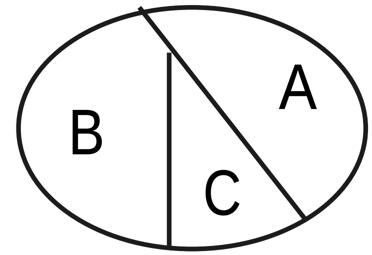
$$H_{\min}(A|B)_\rho := -S_{\max}(\rho_{AB} \| I_A \otimes \rho_B)$$ etc.

analogous to: $S(A|B) = -S(\rho_{AB} \| I_A \otimes \rho_B)$

$$H_{\min}(A:B)_\rho := S_{\min}(\rho_{AB} \| \rho_A \otimes \rho_B)$$ etc.

analogous to: $S(A:B) = S(\rho_{AB} \| \rho_A \otimes \rho_B)$

Min- and Max- Relative Entropies satisfy the:



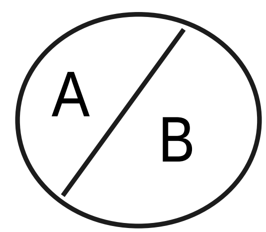
(1) *Strong Subadditivity Property*

just as $$S(\rho_{\mathrm{ABC}}) + S(\rho_{\mathrm{B}}) \le S(\rho_{\mathrm{AB}}) + S(\rho_{\mathrm{BC}})$$

$$H_{\min}(\rho_{\mathrm{ABC}}) + H_{\min}(\rho_{\mathrm{B}}) \le H_{\min}(\rho_{\mathrm{AB}}) + H_{\min}(\rho_{\mathrm{BC}})$$

(2) *Araki-Lieb inequality*

just as $$S(\rho_{\mathrm{AB}}) \ge \left| S(\rho_{\mathrm{A}}) - S(\rho_{\mathrm{B}}) \right|$$



$$H_{\min}(\rho_{\mathrm{AB}}) \ge \left| H_{\min}(\rho_{\mathrm{A}}) - H_{\min}(\rho_{\mathrm{B}}) \right|$$

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(Q) What is the operational significance of the min- and max- relative entropies?

In Quantum information theory, initially one evaluated:

- optimal rates of info-processing tasks:

  e.g. data compression,

  transmission of information through a channel, etc.

  under the following assumptions:

- information sources & channels were memoryless
- they were used an infinite number of times (asymptotic limit) $n \to \infty$

BUT: these assumptions are unrealistic!

In practice :

- Each use of a source or channel need not be memoryless ;

  in fact, correlations/memory effects unavoidable

(e.g. quantum info. source : a highly correlated electron system)

- also sources and channels are used a finite number of times

- Hence it is important to evaluate optimal rates for

  finite number of uses (or even a single use)

  of an arbitrary source or channel

- Corresponding optimal rates:

optimal one-shot rates

Min- & Max relative entropies: $S_{\min}(\rho \| \sigma)$, $S_{\max}(\rho \| \sigma)$

act as *parent quantities* for *one-shot rates* of protocols

just as

Quantum relative entropy: $S(\rho \| \sigma)$

act as a *parent quantity* for *asymptotic rates* of protocols

e.g. Quantum Data Compression

asymptotic rate: $S(\rho) \quad = -S(\rho \| I)$

one-shot rate: $H_{\max}(\rho) = -S_{\min}(\rho \| I)$

[*Koenig & Renner*]

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Further Examples

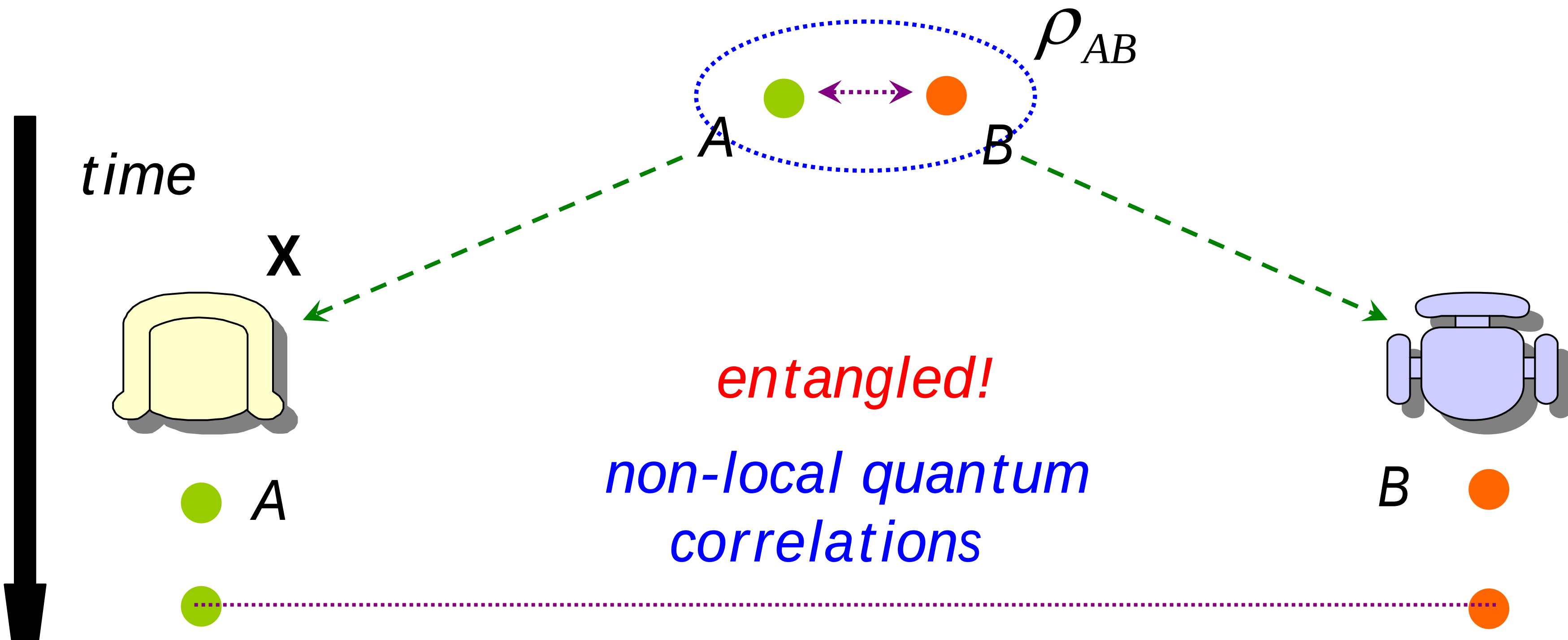
$S_{\min}(\rho \| \sigma)$: *parent quantity for the following:*

- *[Wang & Renner] : one-shot classical capacity of a quantum channel*

- *[ND & Buscemi] : one-shot quantum capacity of a quantum channel*

- Relative entropies → *Entanglement*

"spooky action at a distance"



- *measures here*
- *affected there*

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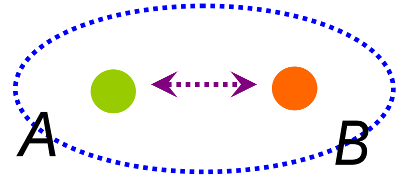
Bipartite Quantum States : $\mathcal{H}_A \otimes \mathcal{H}_B;$



separable

entangled

Separable IF

$$|\Psi_{AB}\rangle = |\vartheta_A\rangle \otimes |\xi_B\rangle$$

else entangled

$$\rho_{AB} = \sum_i p_i \omega_i^A \otimes \sigma_i^B$$

else entangled

Entanglement : *strictly non-local quantum property*

A fundamental feature of entanglement

it cannot be created or increased by :

(i) local operations (LO) & (ii) classical communications(CC)

LOCC

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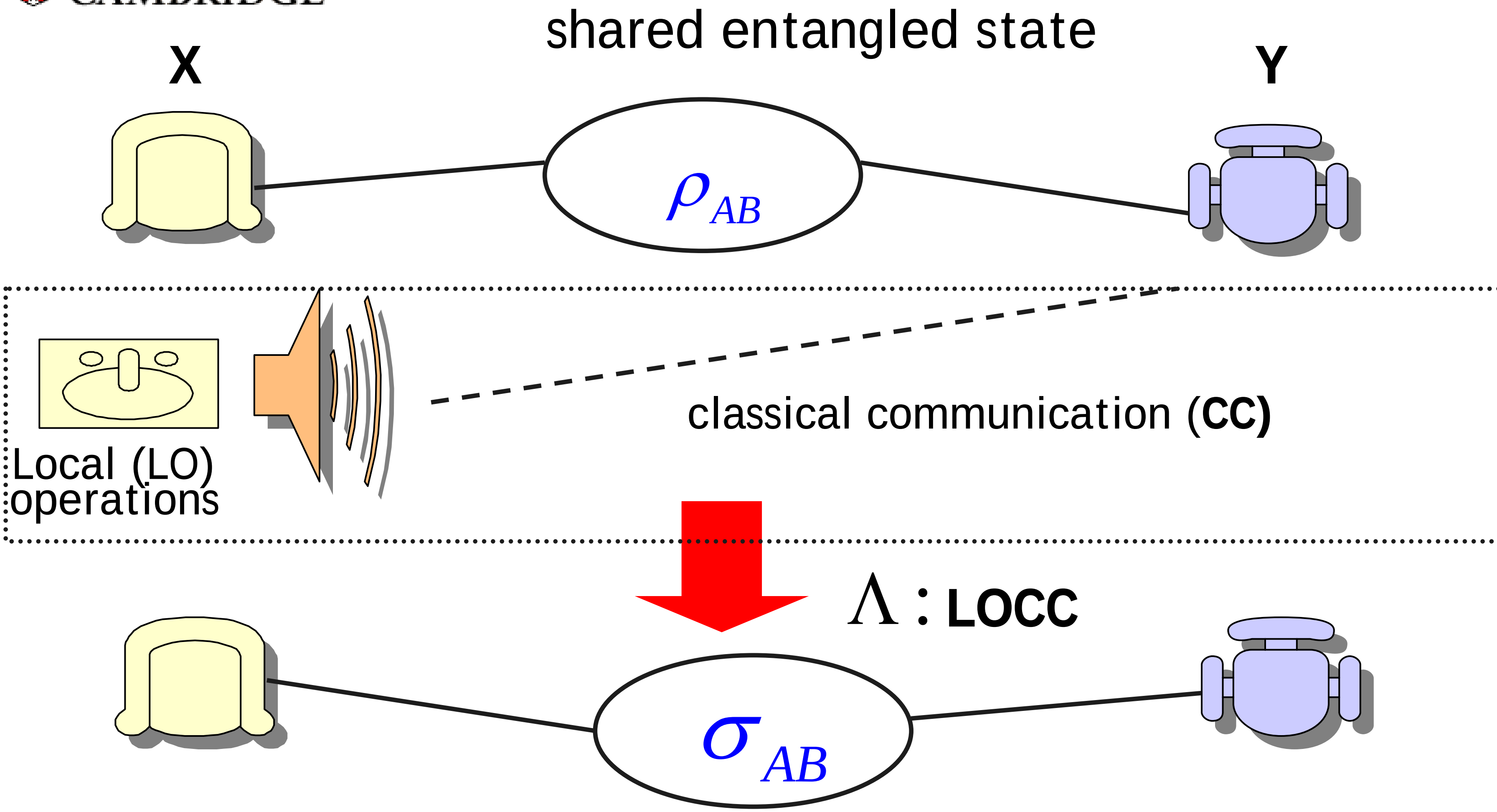

$$\rho_{AB} \xrightarrow[\text{LOCC}]{} \sigma_{AB} \Rightarrow \mathrm{E}(\sigma_{AB}) \leq \mathrm{E}(\rho_{AB})$$

entanglement

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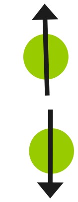
Maximally entangled states

$|0\rangle \equiv$ 
$|1\rangle \equiv$

e.g. Pure state of 2 qubits A,B : $\Psi = |\Psi_{AB}\rangle\langle\Psi_{AB}|$

where

$$|\Psi_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\otimes|0\rangle + |1\rangle\otimes|1\rangle\right)$$

Bell state

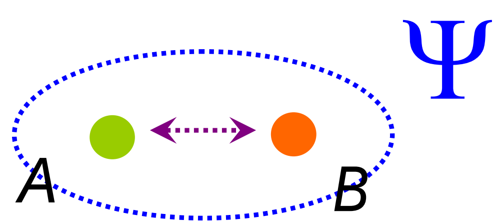
- state of the 2 qubits known exactly ; since AB in pure state
- Reduced state of qubit A

$$\rho_A = \mathrm{Tr}_B \Psi = \frac{I_A}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|$$

completely mixed state

similarly

$$\rho_B = \mathrm{Tr}_A \Psi = \frac{I_B}{2}$$



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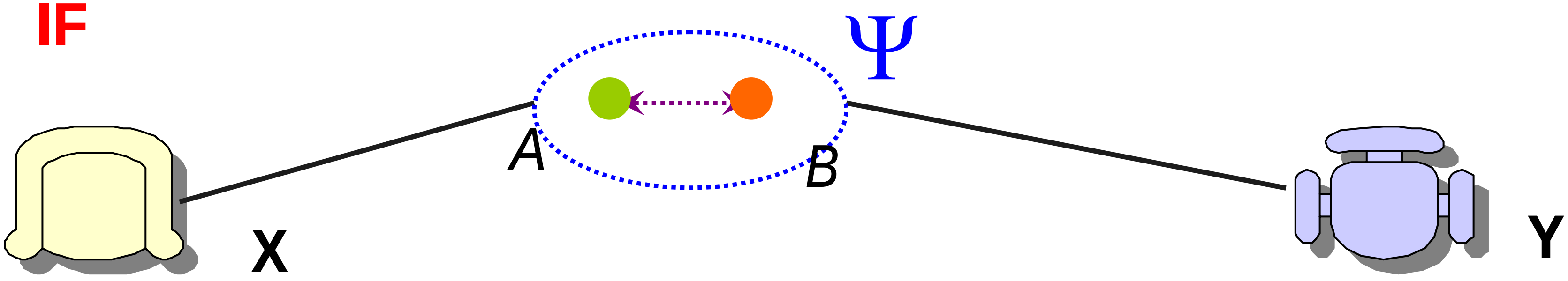
Bell State Ψ

valuable resource in Quantum Info Theory

- **IF**



- They can perform tasks e.g. quantum teleportation

  which are important in the classical realm!

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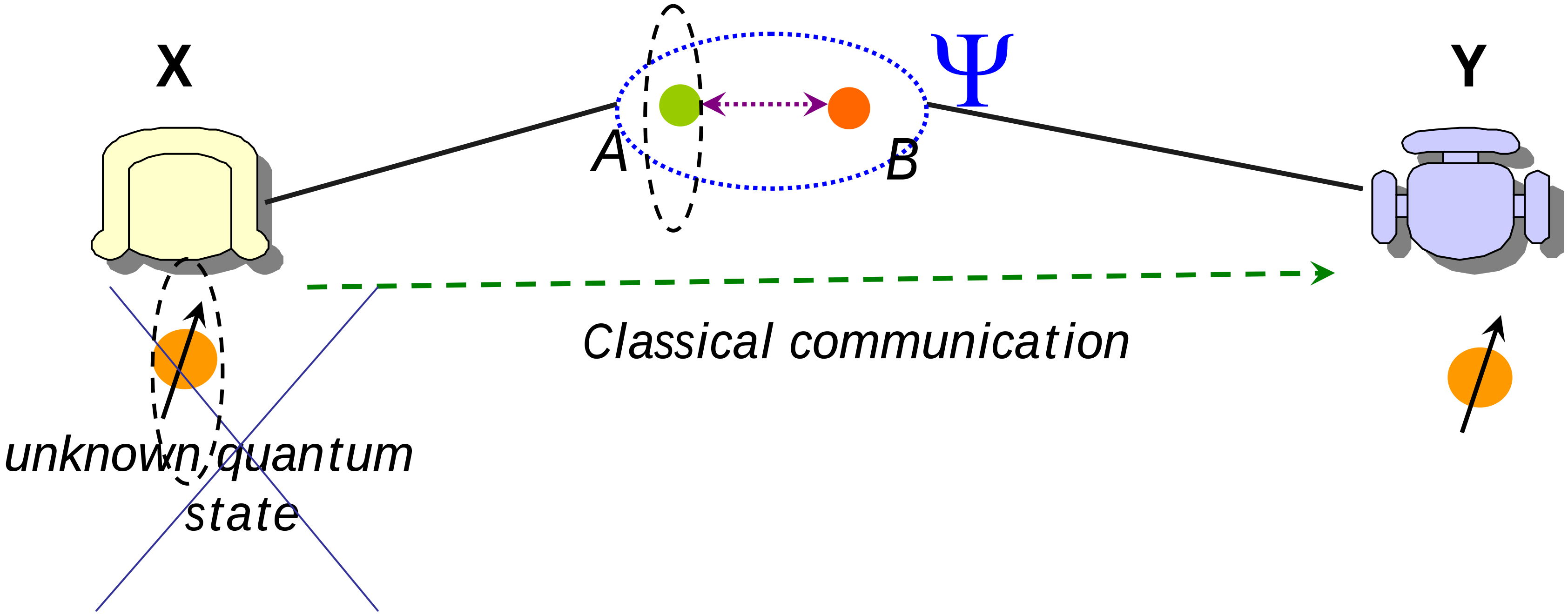
QuantumTeleportation



- *intriguing example of how quantum entanglement can assist practical tasks!*
- *usefulness of Bell states!*

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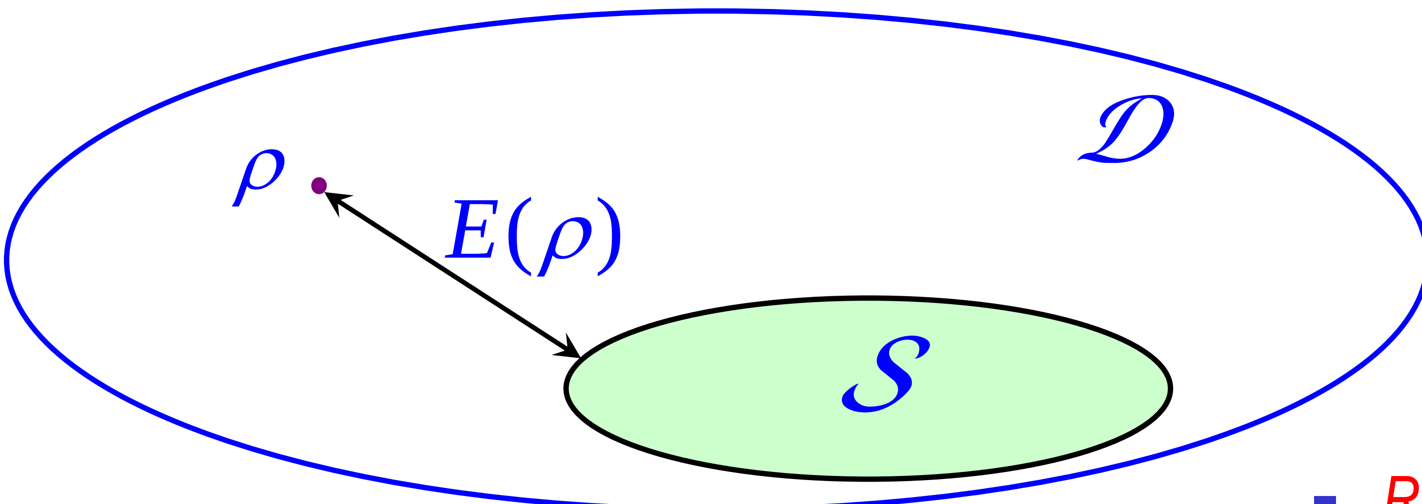
Entanglement monotones

- *A measure of how entangled a state ρ is ;*

  i.e., the amount of entanglement $E(\rho)$ in the state ρ

: "minimum distance" of ρ from the set $\mathcal{S}$ of separable states.



$$E(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho \| \sigma)$$

- *Relative entropy of entanglement*

Properties of an Entanglement Monotone $E(\rho)$

- $E(\rho)=0$ *if* ρ *is separable*
- $E(\Lambda_{\mathrm{LOCC}}(\rho)) \leq E(\rho)$
- $E(\rho)$ is not changed by a local change of basis *etc.*

$$E(\rho)=\min_{\sigma\in\mathcal{S}} S(\rho\,||\,\sigma)$$

satisfies these properties

a valid entanglement monotone

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Entanglement Monotones

$$E(\rho) = \min_{\sigma \in \mathcal{S}} S(\rho \| \sigma)$$ *relative entropy of entanglement*

- We can define two quantities:

$$E_{\max}(\rho) := \min_{\sigma \in \mathcal{S}} S_{\max}(\rho \| \sigma)$$ *Max-relative entropy of entanglement*

$$E_{\min}(\rho) := \min_{\sigma \in \mathcal{S}} S_{\min}(\rho \| \sigma)$$ *Min-relative entropy of entanglement*

these can be proved to be entanglement monotones!

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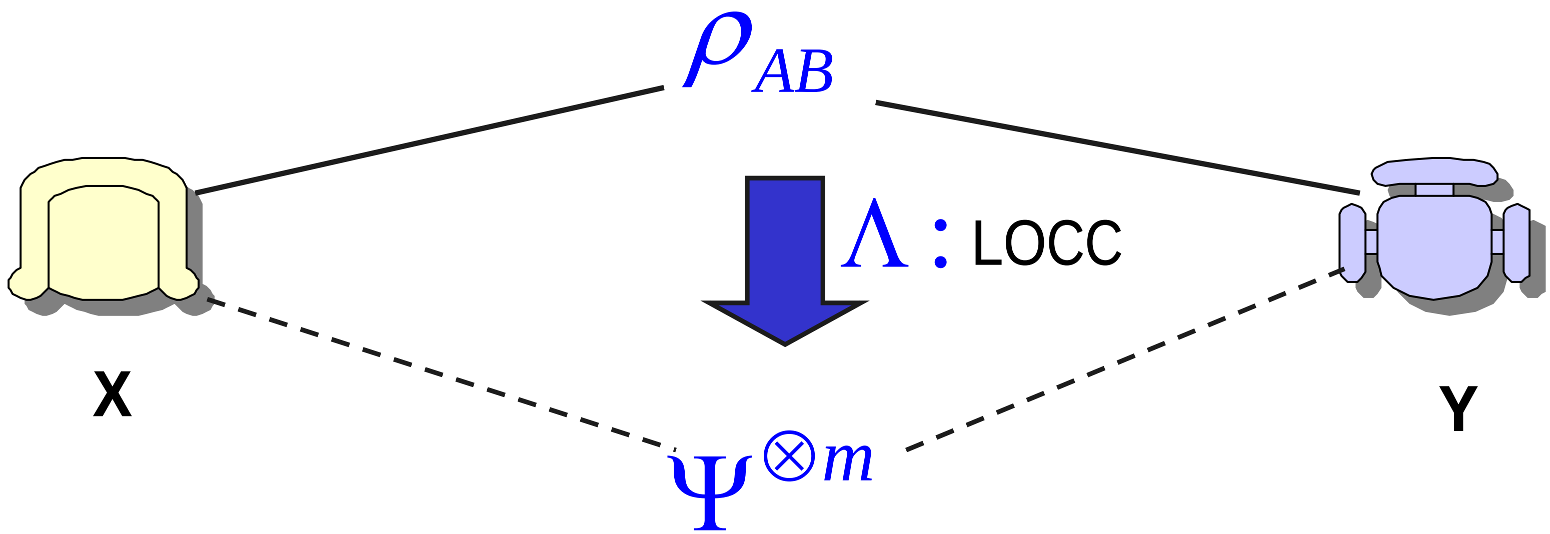
$$E_{\max}(\rho) \quad \text{and} \quad E_{\min}(\rho)$$

have interesting operational significances in

entanglement manipulation

- *What is entanglement manipulation ?*

= Transformation of entanglement from one form to another by **LOCC :**



$\rho_{AB} \in \mathcal{B}(\mathcal{K}_A \otimes \mathcal{K}_B)$

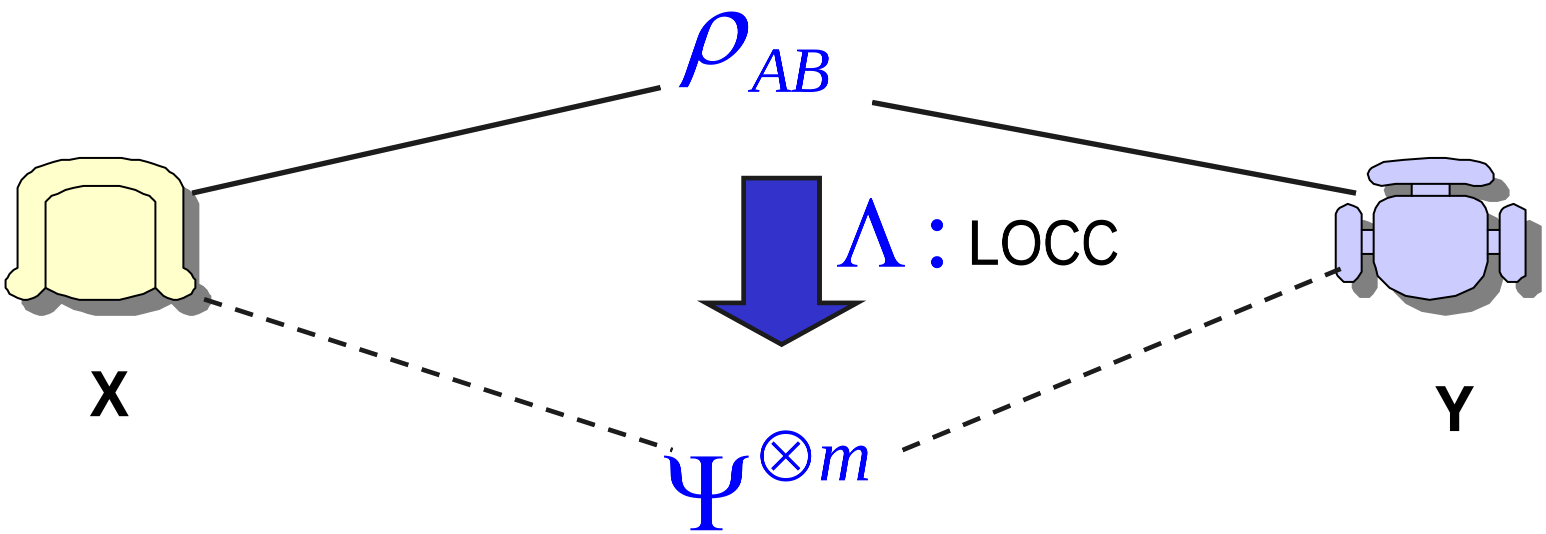
Partially entangled state

$\Psi \in \mathcal{H}_A \otimes \mathcal{H}_B;$

Bell state

$$\dim(\mathcal{K}_A \otimes \mathcal{K}_B) > \dim(\mathcal{H}_A \otimes \mathcal{H}_B)$$

One-Shot Entanglement Distillation



- *What is the maximum number of Bell states that you can extract from the state ρ_{AB} using LOCC?*

i.e., what is the maximum value of m ?

"one-shot distillable entanglement ρ_{AB}"

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- *Result 4 :*

"*one-shot distillable entanglement* of ρ_{AB}"

$$= E_{\min}\left(\rho_{AB}\right)$$

ND &F.Brandao

Min-relative entropy of entanglement

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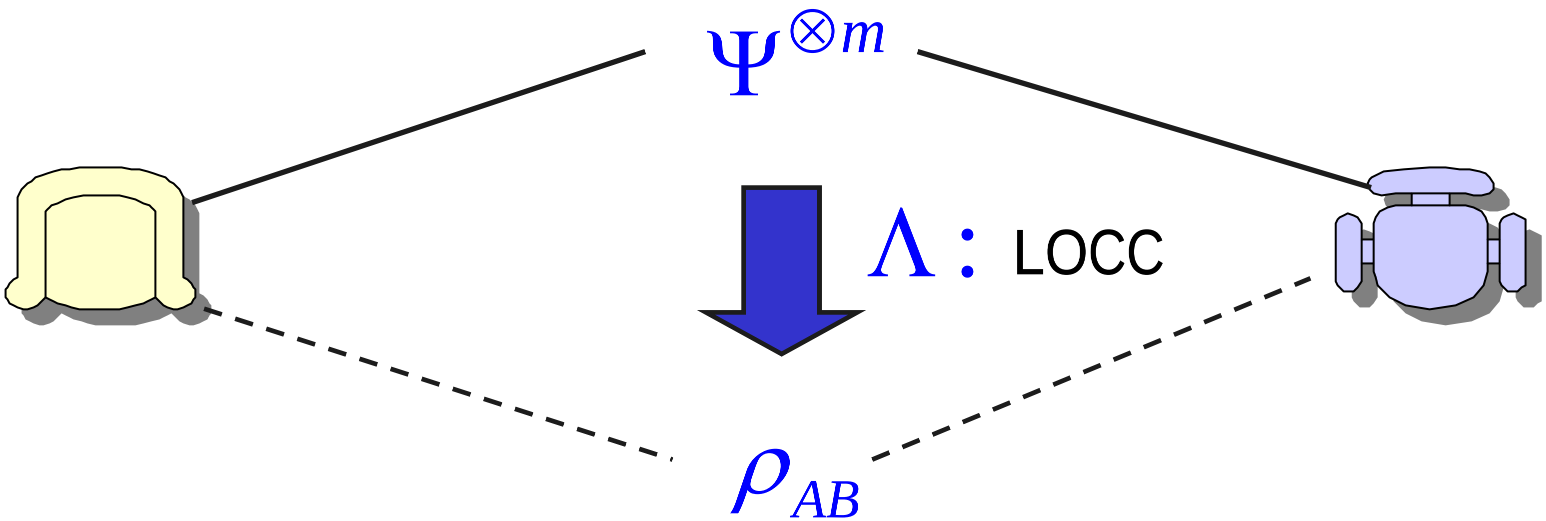
One-shot Entanglement Dilution

- Bell states : resource for creating a desired target state



ρ_{AB} : partially entangled bipartite target state

- *What is the minimum number of Bell states needed to create the state ρ_{AB} using LOCC?*

  i.e., what is the minimum value of m ?

"one-shot entanglement cost of ρ_{AB}"

- *Result 5 :*

"*one-shot entanglement cost of* ρ_{AB}"

$$= E_{\max}\left(\rho_{AB}\right)$$

Max-relative entropy of entanglement

ND &F.Brandao

Summary

- Introduced 2 new relative entropies

  (1) *Min-relative entropy* & (2) *Max-relative entropy*

$$S_{\min}(\rho\|\sigma) \leq S(\rho\|\sigma) \leq S_{\max}(\rho\|\sigma)$$

- *Parent quantities for optimal one-shot rates for*
  - (i) data compression for a quantum info source
  - (ii) transmission of (a) classical info & (b) quantum info

    through a quantum channel

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Entanglement monotones

- *Min-relative entropy of entanglement* $E_{\min}(\rho_{AB})$
- *Max-relative entropy of entanglement* $E_{\max}(\rho_{AB})$
- *Operational interpretations:*

$E_{\min}(\rho_{AB})$: *One-shot distillable entanglement of* ρ_{AB}

$E_{\max}(\rho_{AB})$: *One-shot entanglement cost of* ρ_{AB}

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- F. Brandao & ND: *One-shot entanglement manipulation under non-entangling maps (to appear in IEEE Trans. Inf. Thy, 2009)*

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