Classical and Quantum Localization in two and three dimensions

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This talk is about some mathematical results on physical models of particle transport in random media

- propagation in a random medium is often modelled by a network:
- medium represented by some fixed graph $\mathcal{G}$ (initially closed)
  - for convenience the edges of $\mathcal{G}$ are assumed *oriented*, and each node has exactly 2 incoming and 2 outgoing edges

- at each tick of the clock, the particle moves through a single node, from an edge $e$ to a neighbouring edge $e'$
Classical version

- decompose each node at which \((e_1 \cup e_2) \rightarrow (e'_1 \cup e'_2)\) either into \((e_1 \rightarrow e'_1) \cup (e_2 \rightarrow e'_2)\) with probability \(p(e_1, e'_1) = p(e_2, e'_2)\); or into \((e_1 \rightarrow e'_2) \cup (e_2 \rightarrow e'_1)\) with probability \(p(e_1, e'_2) = p(e_2, e'_1) = 1 - p(e_1, e'_1)\)

- this decomposes \(\mathcal{G}\) into a union of cycles (closed loops), so we have a deterministic walk in a random medium

- equivalently, do not decompose \(\mathcal{G}\), but let the particle, the first time it reaches a given node along (say) edge \(e_1\), choose to exit via \(e'_j\) \((j = 1, 2)\) with probability \(p(e_1, e'_j)\)

- the next time it visits the node, it either approaches along \(e_1\) in which case it must exit the same way as it did before, or it approaches along \(e_2\), in which it must exit the other way

- this is a history-dependent random walk
Quantum version

- there is an $N$-dimensional vector space $\mathcal{H}_e$ on each edge
- instead of real-valued probabilities $p(e, e')$ we have amplitudes $S(e', e)$, which are linear maps $\mathcal{H}_e \rightarrow \mathcal{H}_{e'}$ with $\sum_{e'} S^\dagger(e, e')S(e', e) = 1$
- in addition, the state vector of the particle can get rotated by a unitary operator $U_e : \mathcal{H}_e \rightarrow \mathcal{H}_e$ as it propagates along a given edge $e$
- the quantum amplitude for the particle to propagate from $e_1$ to $e_2$ is a linear map from $\mathcal{H}_{e_1} \rightarrow \mathcal{H}_{e_2}$ given by a sum over random walks (Feynman paths $\Gamma$), each weighted by a product of maps along the path $\Gamma = (e_1, e'_1, \ldots, e_2)$:

$$G(e_2, e_1) = \sum_{\Gamma} U_{e_2}^{1/2} \cdots U(e'_1)S(e'_1, e_1)U_{e_1}^{1/2}$$
a point conductance measurement corresponds to breaking open a subset \( \{ e \} \) of edges of \( \mathcal{G} \), \( e \rightarrow (e_{\text{in}}, e_{\text{out}}) \), choosing a pair of these \( (e_{\text{in},1}, e_{\text{out},2}) \) and computing

\[
\text{Tr } G(e_{\text{in},1}, e_{\text{out},2})^\dagger G(e_{\text{out},2}, e_{\text{in},1})
\]

the off-diagonal terms between the paths contributing to \( G^\dagger \) and \( G \) produce quantum interference effects

in principle the amplitudes \( S(e', e) \) and \( U_e \) are fixed, but we may view these as drawn from some random ensemble

- e.g. we can choose the \( U_e \) to be independent random matrices drawn from the invariant measure on \( \text{SU}(N) \)

self-averaging: in the thermodynamic limit \( \vert \mathcal{G} \vert \rightarrow \infty \) many physically interesting quantities are almost surely equal to their mean values
for the case $N = 1$ the $U_e$ are pure phases $e^{i\phi_e}$ and $S(e, e')$ are complex numbers, but actually this is harder than the case $N = 2$, where we have the

**Theorem.** The mean point conductance $\text{Tr} \, G(e_{in}, e_{out})^\dagger G(e_{out}, e_{in})$ is given by the probability that there exists an open path from $e_{in}$ to $e_{out}$ in the classical model with $p(e, e') = (1/N) \text{Tr} \, S(e, e')^\dagger S(e', e)$. The proof is either graph-theoretic (Beamond, Chalker, JC 2002) or using a supersymmetric formalism (JC 2005) but in both case depends on a special properties of the invariant measure on SU(2) matrices:

$$\int U^p \, dU = \begin{cases} 
1 & p = 0 \\
-\frac{1}{2} & p = 2 \\
0 & \text{otherwise}
\end{cases}$$
the interesting cases are when $\mathcal{G}$ is a regular lattice in $\mathbb{R}^d$, in the limit when $|\mathcal{G}| \to \infty$:

- if the loops in the decomposition of $\mathcal{G}$ are a.s. of finite length as $|\mathcal{G}| \to \infty$, the states in the corresponding problem are all localized.

- on the other hand, if there is a finite probability of escape to infinity in the classical problem, the quantum states are extended.

- since classical problems are easier than quantum ones, studying the former may tell us something about the latter.

- the physics literature suggests that
  
  - for $d = 2$ states are usually localised except in special cases where an extra symmetry holds, and in $d = 3$ there can be a transition from localized to extended states
  
  - in the extended phase, there should be diffusive behaviour, that is the classical paths should scale to Brownian motion
  
  - at the transition in $d = 3$, and in the special cases in $d = 2$, one expects power laws in, for example, the distribution of loop lengths
A 2d special case: the L-lattice

- decompose each node according to

  - each decomposition of $G$ corresponds to a bond percolation configuration on a $\mathbb{Z}^2$ lattice
  - the loops are the external and internal hulls of percolation clusters
  - if $p \neq p_c = \frac{1}{2}$ almost all the loops are finite as $|G| \to \infty$
  - at $p = \frac{1}{2}$ there is a power-law distribution of loop lengths
  - corresponding quantum problem is the SU(2) version of the network model for the integer quantum Hall plateau transition (Chalker, Coddington 1988); the connection to bond percolation was first realised by Gruzberg et al. 1999.
A more generic case: the Manhattan lattice

- For $p > \frac{1}{2}$ all loops are trapped in finite voids within percolation clusters and are therefore finite.
- In fact, analytic RG calculations and simulations suggest that loops are always finite as long as $p > 0$ (Beamond, Owczarek, JC 2003), consistent with the idea that generically all states are localized in $d = 2$, but there is no proof (Sidoravicius et al. have a proof for a variant of this model.)
Three dimensions

- how to build a regular lattice in $\mathbb{R}^3$ with coordination number 4 and cubic symmetry:

- start with the faces of $\mathbb{Z}^3$
add the faces of \((\mathbf{Z} + \frac{1}{2})^3\):
\( \mathcal{G} \) is the intersection of the two sets of faces
bond percolation on one sublattice of $\mathbb{Z}^3$ (an fcc lattice) decomposes the faces of $(\mathbb{Z} + \frac{1}{2})^3$ into hulls $\{H\}$: closed oriented surfaces which separate the clusters from the dual clusters
similarly, bond percolation on one sublattice of \((\mathbb{Z} + \frac{1}{2})^3\) has hulls \(\{H'\}\) composed of faces of \(\mathbb{Z}^3\)

\(\{H\} \cap \{H'\}\) gives a decomposition of \(\mathcal{G}\)
these paths are therefore a.s. finite if \( p < p_c^{\text{fcc}} \approx 0.12 \)

this problem has a symmetry under \( p \to 1 - p \); simulations suggest that \( p = \frac{1}{2} \) is in the extended phase, but no proof

in the Manhattan version of this problem, paths are localised for \( p > 1 - p_c^{\text{fcc}} \), and we expect them to be asymptotically like random walks for small \( p \), but no proof yet
Summary

- random decompositions of graphs into the union of cycles give rise to interesting models of both classical and quantum localisation

Some references: