Classical and Quantum Localization in two and three dimensions

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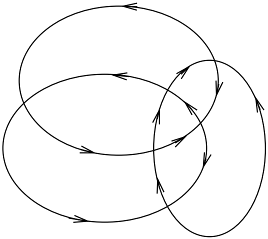
This talk is about some mathematical results on physical models of particle transport in random media

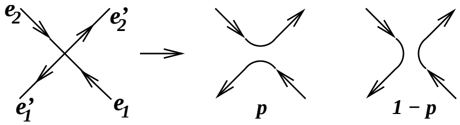
- propagation in a random medium is often modelled by a *network*:
- medium represented by some fixed graph $\mathcal{G}$ (initially closed)
  - for convenience the edges of $\mathcal{G}$ are assumed *oriented*, and each node has exactly 2 incoming and 2 outgoing edges



- at each tick of the clock, the particle moves through a single node, from an edge e to a neighbouring edge e'

Classical version

- decompose each node at which $(e_1 \cup e_2) \to (e'_1 \cup e'_2)$ either into $(e_1 \to e'_1) \cup (e_2 \to e'_2)$ with probability $p(e_1, e'_1) = p(e_2, e'_2)$; or into $(e_1 \to e'_2) \cup (e_2 \to e'_1)$ with probability $p(e_1, e'_2) = p(e_2, e'_1) = 1 - p(e_1, e'_1)$



- this decomposes $\mathcal{G}$ into a union of cycles (closed loops), so we have a *deterministic* walk in a *random* medium
- *equivalently*, do not decompose $\mathcal{G}$, but let the particle, the first time it reaches a given node along (say) edge e_1, choose to exit via e'_j $(j = 1, 2)$ with probability $p(e_1, e'_j)$
- the next time it visits the node, it either approaches along e_1 in which case it *must* exit the same way as it did before, or it approaches along e_2, in which it must exit the other way
- this is a *history-dependent random* walk

Quantum version

- there is an N-dimensional vector space $\mathcal{H}_e$ on each edge
- instead of real-valued probabilities $p(e, e')$ we have *amplitudes* $S(e', e)$, which are linear maps $\mathcal{H}_e \to \mathcal{H}_{e'}$ with $\sum_{e'} S^\dagger(e, e')S(e', e) = 1$
- in addition, the state vector of the particle can get rotated by a unitary operator $U_e : \mathcal{H}_e \to \mathcal{H}_e$ as it propagates along a given edge e
- the quantum amplitude for the particle to propagate from e_1 to e_2 is a linear map from $\mathcal{H}_{e_1} \to \mathcal{H}_{e_2}$ given by a sum over random walks (Feynman paths Γ), each weighted by a product of maps along the path $\Gamma = (e_1, e_1', \dots, e_2)$:

$$G(e_2, e_1) = \sum_{\Gamma} U_{e_2}^{1/2} \cdots U(e_1')S(e_1', e_1)U_{e_1}^{1/2}$$

- a *point conductance* measurement corresponds to breaking open a subset $\{e\}$ of edges of $\mathcal{G}$, $e \to (e_{\text{in}}, e_{\text{out}})$, choosing a pair of these $(e_{\text{in},1}, e_{\text{out},2})$ and computing

$$\operatorname{Tr} G(e_{\text{in},1}, e_{\text{out},2})^{\dagger} G(e_{\text{out},2}, e_{\text{in},1})$$

- the off-diagonal terms between the paths contributing to $G^{\dagger}$ and G produce *quantum interference* effects
- in principle the amplitudes $S(e', e)$ and U_e are fixed, but we may view these as drawn from some random ensemble
  - e.g. we can choose the U_e to be independent random matrices drawn from the invariant measure on $\mathrm{SU}(N)$
- *self-averaging*: in the thermodynamic limit $|\mathcal{G}| \to \infty$ many physically interesting quantities are almost surely equal to their mean values

- for the case $N = 1$ the U_e are pure phases $e^{i\phi_e}$ and $S(e, e')$ are complex numbers, but actually this is harder than the case
- $N = 2$, where we have the

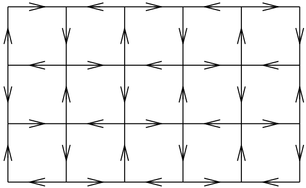
Theorem. *The mean point conductance* $\mathrm{Tr}\,\overline{G(e_{\mathrm{in}}, e_{\mathrm{out}})^\dagger G(e_{\mathrm{out}}, e_{\mathrm{in}})}$ *is given by the probability that there exists an open path from* e_{in} *to* e_{out} *in the classical model with* $p(e, e') = (1/N)\mathrm{Tr}\, S(e, e')^\dagger S(e', e)$.

The proof is either graph-theoretic (Beamond, Chalker, JC 2002) or using a supersymmetric formalism (JC 2005) but in both case depends on a special properties of the invariant measure on SU(2) matrices:

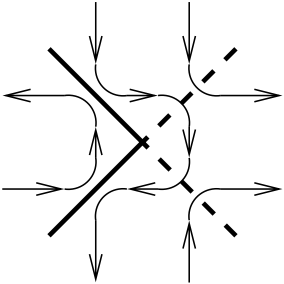
$$\int U^p \, dU = \begin{cases} 1 & p = 0 \\ -\frac{1}{2} & p = 2 \\ 0 & \text{otherwise} \end{cases}$$

- the interesting cases are when $\mathcal{G}$ is a regular lattice in $\mathbf{R}^d$, in the limit when $|\mathcal{G}| \to \infty$:
- if the loops in the decomposition of $\mathcal{G}$ are a.s. of finite length as $|\mathcal{G}| \to \infty$, the states in the corresponding problem are all *localized*.
- on the other hand, if there is a finite probability of escape to infinity in the classical problem, the quantum states are *extended*.
- since classical problems are easier than quantum ones, studying the former may tell us something about the latter
- the physics literature suggests that
  - for $d = 2$ states are usually localised except in special cases where an extra symmetry holds, and in $d = 3$ there can be a transition from localized to extended states
  - in the extended phase, there should be diffusive behaviour, that is the classical paths should scale to Brownian motion
  - at the transition in $d = 3$, and in the special cases in $d = 2$, one expects power laws in, for example, the distribution of loop lengths

A 2d special case: the L-lattice

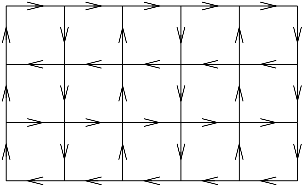


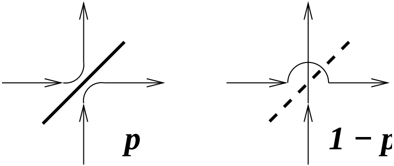
- decompose each node according to



- each decomposition of $\mathcal{G}$ corresponds to a bond percolation configuration on a $\mathbf{Z}^2$ lattice
- the loops are the external and internal hulls of percolation clusters
- if $p \neq p_c = \frac{1}{2}$ almost all the loops are finite as $|\mathcal{G}| \to \infty$
- at $p = \frac{1}{2}$ there is a power-law distribution of loop lengths
- corresponding quantum problem is the $\mathrm{SU}(2)$ version of the network model for the integer quantum Hall plateau transition (Chalker, Coddington 1988); the connection to bond percolation was first realised by Gruzberg *et al.* 1999.

A more generic case: the Manhattan lattice

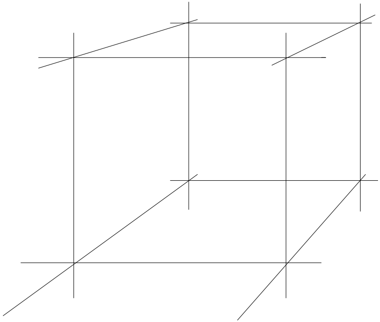
- 



- for $p > \frac{1}{2}$ all loops are trapped in finite voids within percolation clusters and are therefore finite
- in fact, analytic RG calculations and simulations suggest that loops are always finite as long as $p > 0$ (Beamond, Owczarek, JC 2003), consistent with the idea that generically all states are localized in $d = 2$, but there is no proof (Sidoravicius *et al.* have a proof for a variant of this model.)

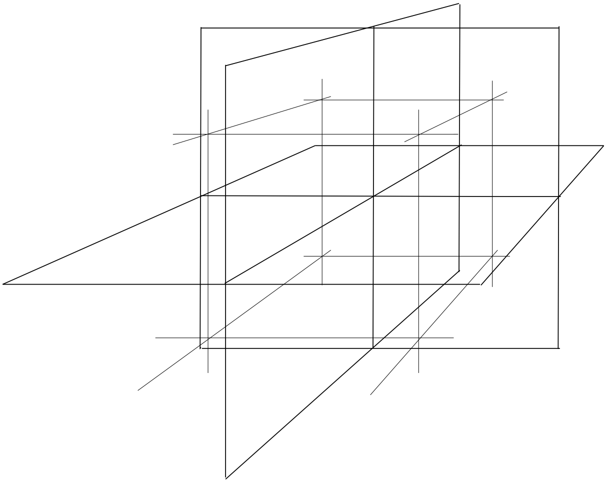
Three dimensions

- how to build a regular lattice in $\mathbf{R}^3$ with coordination number 4 and cubic symmetry:

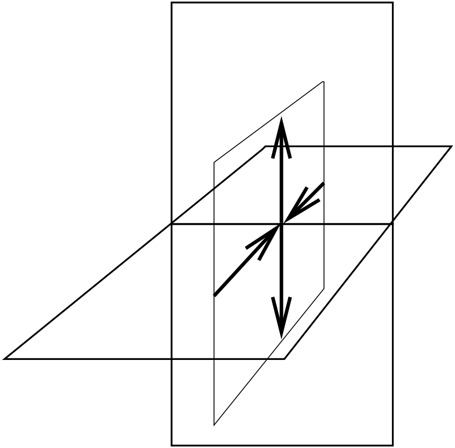


- start with the faces of $\mathbf{Z}^3$

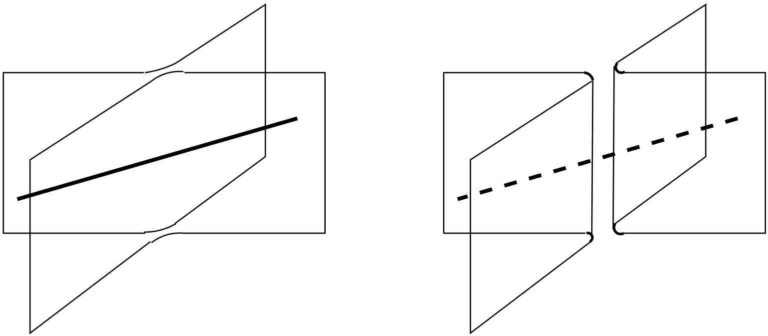
- add the faces of $(\mathbf{Z}+\frac{1}{2})^3$:

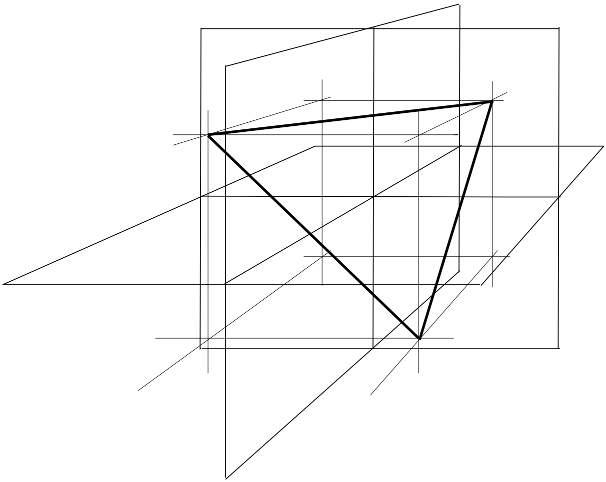


- $\mathcal{G}$ is the intersection of the two sets of faces

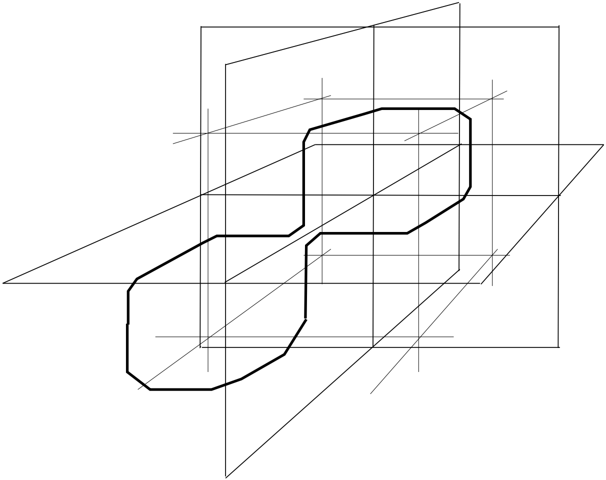


- bond percolation on one sublattice of $\mathbf{Z}^3$ (an fcc lattice) decomposes the faces of $(\mathbf{Z}+\frac{1}{2})^3$ into hulls $\{H\}$: closed oriented surfaces which separate the clusters from the dual clusters





- similarly, bond percolation on one sublattice of $(\mathbf{Z}+\frac{1}{2})^3$ has hulls $\{H'\}$ composed of faces of $\mathbf{Z}^3$
- $\{H\} \cap \{H'\}$ gives a decomposition of $\mathcal{G}$



- these paths are therefore a.s. finite if $p < p_c^{\text{fcc}} \approx 0.12$
- this problem has a symmetry under $p \to 1 - p$; simulations suggest that $p = \frac{1}{2}$ is in the extended phase, but no proof
- in the Manhattan version of this problem, paths are localised for $p > 1 - p_c^{\text{fcc}}$, and we expect them to be asymptotically like random walks for small p, but no proof yet

Summary

- random decompositions of graphs into the union of cycles give rise to interesting models of both classical and quantum localisation

Some references:

- L-lattice: Gruzberg et al., Phys.Rev.Lett.82:4524, 1999
- main theorem for general graphs: Beamond, JC, Chalker, Phys. Rev. B. 65, 214301, 2002; JC Comm. Math. Phys. 1432-0916, 2005
- Manhattan lattice: Beamond, JC, Owczarek, J. Phys. A 36, 10251, 2003
- 3d lattices: Ortuno, Somoza, Chalker, Phys.Rev.Lett. 102, 070603, 2009; JC + Y Ikhlef, in preparation.