Self-avoiding walk and the renormalisation group

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Constructive and Multiscale Methods in Quantum Theory

Self-avoiding walks on $\mathbb{Z}^d$ are simple-random walk paths without self-intersections. Self-avoiding walks of the same length are declared to be equally likely. The basic question is how far on average is their endpoint from the origin?
After a brief review of the current state of knowledge I will describe work in progress with Gordon Slade for the case $d = 4$ which is an application of the renormalisation group to a supersymmetric lattice field theory. Our immediate goal is to prove that the critical two-point function (Green function) for a spread-out model of self-avoiding walks on $\mathbb{Z}^d$ decays like $|x|^{-2}$ at large distances, as it does for simple random walk.

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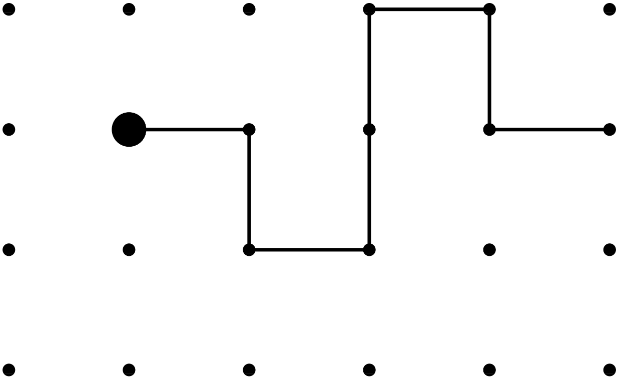

Figure: An $N = 8$ step self-avoiding walk

Equal probability for all N step self-avoiding walks.

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Defn: $a_N \sim b_N$ if $\lim_{N\to\infty} \frac{a_N}{b_N} \in (0, \infty)$.

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Brydges-Imbrie 2003: true for weak coupling Edwards model on hierarchical four dimensional lattice

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- At $\lambda = \lambda_c$, $G_{a,b} \sim |a-b|^{-(d-2)}$

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- For $g \geq 0$ there exists $\lambda_c = \lambda_c(g)$ s.t. divergent for $\lambda < \lambda_c$, convergent for $\lambda > \lambda_c$

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- The last step is an abelian version of
  $\mathbb{E}\big(\text{end-to-end distance }\big) \sim N^{1/2}\log^{1/8}(N)$

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- D. Iagolnitzer and J. Magnen 1994: Continuum Edwards Model:

$$\mathrm{Gr}_{ab}(z) \sim |a-b|^{-2} \left(1 + \lambda_1 \frac{1}{\ln|a-b|} + \lambda_2 \frac{\ln\ln|a-b|}{\ln^2|a-b|} + r(a,b)\right)$$

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- Edwards model based on Lévy walk on $\mathbb{Z}^3$ (model has upper critical dimension $3+\epsilon$): Mitter, Scoppola (2008).

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- V_x = replace ϕ by ϕ_x.
- $V(\Lambda) = \sum_{x\in\Lambda} V_x$.
- There exist C and V such that

$$G_{a,b} = \lim_{\Lambda\uparrow\mathbb{Z}^4} \mathbb{E}\left(e^{-V(\Lambda)}\bar{\varphi}_a\varphi_b\right)$$

References

References

- Parisi-Sourlas (1980); McKane, (1980); Luttinger (1983);

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- Review: Brydges, Imbrie, Slade (2009).

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Brydges, Guadagni, Mitter (2004): existence for periodic lattice $C = (\epsilon - \Delta)^{-1}$ for $\epsilon > 0$.

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- where, on the right hand side,

$$\phi = \xi_1 + \xi_2 + \cdots + \xi_N$$

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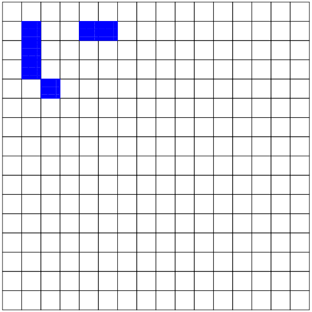
- $Z_N = \mathbb{E} Z_0$ is our Green's function $G^{\Lambda}_{a,b}$

Lattice Geometry



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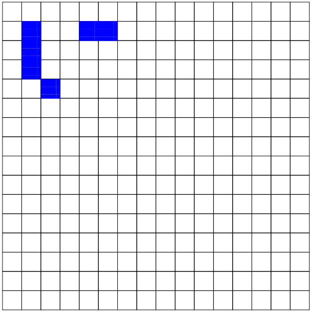

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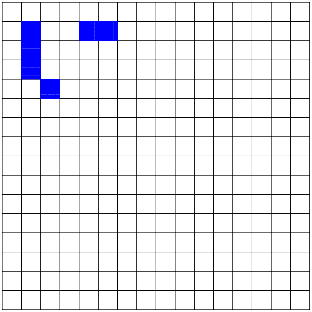

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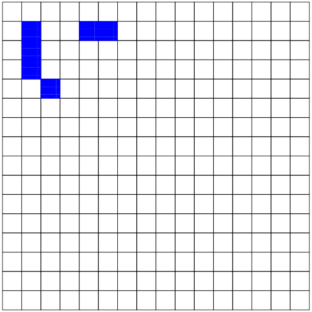

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- Given functions I, K defined on $\mathcal{P}_j$, let

$$(I \circ K)(\Lambda) = \sum_{X \in \mathcal{P}_j} I(\Lambda \setminus X) K(X)$$

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- What about $\bar{\varphi}_a \varphi_b$? Add $\sigma_a \bar{\varphi}_a + \bar{\sigma}_b \varphi_b$ to $V_0(\Lambda)$ so that

$$G_{a,b} = \frac{\partial}{\partial \sigma_a} \frac{\partial}{\partial \bar{\sigma}_b}\bigg|_{\sigma=0} \mathbb{E}\, I_0 \circ K_0(\Lambda)$$

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- Given (I_j, K_j) there exist (I_{j+1}, K_{j+1}) such that

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- $V_j \equiv (g_j, a_j, z_j, \sigma_j)$.

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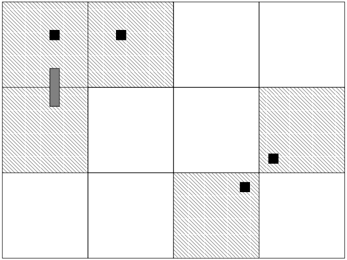

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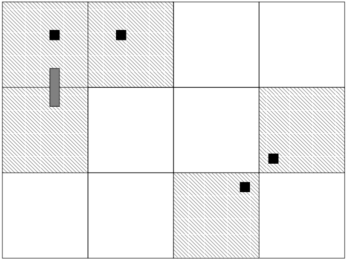

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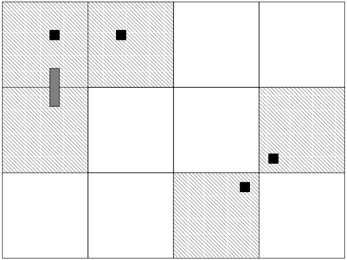

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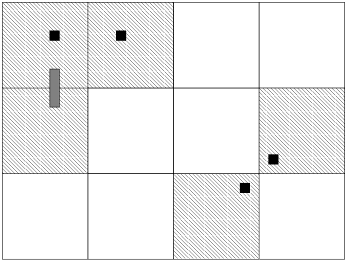

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- Entropy: For $U \in \mathcal{P}_{j+1}$, let $\overline{\mathcal{P}}_j(U) = \{X \in \mathcal{P}_j : \overline{X} = U\}$.

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- with $\tilde{K}$ given by: for $U \in \mathcal{P}_{j+1}$,

$$\tilde{K}(U) = \sum_{X \in \overline{\mathcal{P}}_j(U)} I_{j+1}(U \setminus X)\mathbb{E}_{j+1}(K_j \circ \delta I)(X).$$

Picture illustrating $\tilde{K}(U)$

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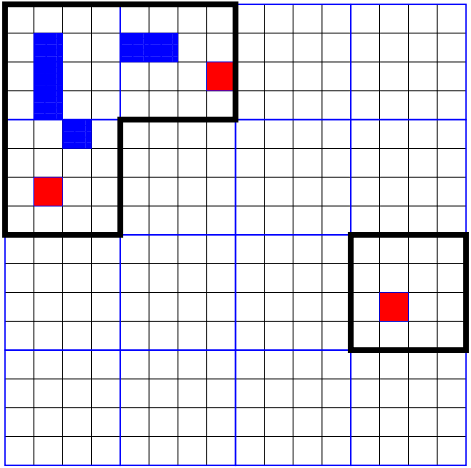

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- Summary: Factorisation of K_j at scale $j \Rightarrow$ Factorisation of $\tilde{K}$ at scale $j+1$

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- Example: $\sum_{X:X\supset B} \|K(X)\| \ll \|V(B)\|$ for all $B \in \mathcal{B}_j$.
- Such estimates cannot survive an induction $j \to j+1$ unless V_{j+1} is chosen well

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  - $\sigma_{j+1} = \sigma_j + r_\sigma(\xi)$
- $z_{j+1} = \sum_{k \geq j} r_{z,k}(\xi)$ is summable

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- $\Rightarrow G_{a,b} \sim |a-b|^{-2}$