Quantum-classical point processes.

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**General situation**

$(X, \mathcal{B}, \mathcal{B}_0)$ is a general phase space. $\mathfrak{X} = \mathcal{M}_f(X)$ denotes the collection of configurations in $X$, i.e. the finite point measures in $X$.

The point of departure is a complex measure $L$ on $\mathfrak{X}$ satisfying $L\{o\} = 0$. It is the generator of the main object

$$\mathcal{S}_L = \exp (L - L(1)\Delta o).$$

Here the right hand side is defined as $\exp K = \sum_{n=0}^{\infty} \frac{1}{n!} K^n$. $\mathcal{S}_L$ is a complex measure which is normalized in the sense $\mathcal{S}_L 1 = 1$, and has the representation

$$(1) \quad \mathcal{S}_L = \frac{1}{\Xi} \exp L, \quad \Xi = \exp L 1.$$
We are only interested in the case where $\mathcal{S}_L$ is a law, i.e. a probability $P$ on $X$, i.e. a finite point process in $X$.

In typical situations a canonical procedure allows to extend these laws to processes with infinitely many particles.

**Interpretation of $\mathcal{S}_L$** According to (1) $\mathcal{S}_L$ then realizes finite configurations of particles in $X$ which are the result of finitely many independent superpositions of finite clusters (which are generated by $L$).

**A special class of $L$: Quantum processes**

We are given a *quantum interaction* $\kappa$, i.e. a Hermitean kernel $\kappa$ of positive type ($\kappa$ positive definit) together with $\tau > 0$ and
\( \varepsilon \in \{-1,+1\} \). We then consider the complex pseudomeasure

\[
\mathcal{L} \varphi = \sum_{m=1}^{\infty} \varepsilon^{m-1} z^m \int_{X^m} \varphi(\delta x_1 + \cdots + \delta x_m) \kappa(x_1, x_2) \kappa(x_2, x_3) \cdots \kappa(x_m, x_1) \, d x_1 \cdots d x_m.
\]

(2)

Here and in the sequel \( \lambda(dx) = dx \) denotes a locally finite reference measure in the phase space. Under general and natural additional integrability assumptions on \( \kappa \) one then can show for \( z \) small enough the existence of an infinitely extended point process \( P = \mathcal{S}_\mathcal{L} \) in \( X \), which is locally specified by the measures

\[

\varrho_k(dx_1 \cdots dx_k) = z^k \varphi_\varepsilon^\kappa(\delta x_1 + \cdots + \delta x_k) \, dx_1 \cdots dx_k.
\]

(3)

The density \( \varphi_\varepsilon^\kappa \) is the immanant of \( (\kappa, \varepsilon) \); i.e. in the case \( \varepsilon = -1 \) the determinant \( \det(\kappa(x_i, x_j))_{i,j=1}^{k} \) and if \( \varepsilon = +1 \) the permanent of this matrix. This process \( P = \mathcal{S}_\mathcal{L} \) we call quantum process specified by \( (\kappa, \varepsilon) \).
One can show that $L$ is of first order, and thereby $P$ satisfies the so-called \textit{cluster equation}

\begin{equation}
C_P = C_L \ast P,
\end{equation}

which in some sense replaces the DLR-equation in the case of classical systems. Here $C_P$ denotes the \textit{Campbell measure of $P$} which contains the information of all moment measures of $P$. The operation $\ast$ is some version of a convolution.

The cluster equation immediately implies that the intensity of $P$ is

$$\nu^1_P(dx) = \nu^1_L(dx) = K_\kappa^\varepsilon(x,x). dx$$

Here

$$K_\kappa^\varepsilon = \sum_{m=1}^{\infty} \varepsilon^{m-1} z^m k^\ast m$$
is the so called \textit{correlation kernel for} $\kappa$.

Consider the factorial moment measures $\hat{\nu}_P^k$ of order $k \geq 2$ of $P$. They are defined by

$$\hat{\nu}_P^k f = \int_{\mathcal{M}^k(X)} \hat{\mu}^k(f) P(d\mu), \quad f \in \mathcal{K}(X).$$

Here

$$\hat{\mu}^k(dx_1 \ldots dx_k) = \mu(dx_1)(\mu - \delta x_1)(dx_2) \cdots \left(\mu - \sum_{j=1}^{k-1} \delta x_j\right)(dx_k).$$

One can show that they are dominated by $\lambda^k$:

(5) \quad $\hat{\nu}_P^k(dx_1 \ldots dx_k) = \varphi_{\kappa}^{K_\varepsilon}(\delta x_1 + \cdots + \delta x_k).dx_1 \ldots dx_k.$

The immanant appearing as a density is the correlation function of $P$. Equation (5) signifies that $P$ is an immanantal process for
$K^\varepsilon_\kappa$, thus a permanental process if $\varepsilon = +1$ and a determinantal process if $\varepsilon = -1$.

To summarize:

**Proposition 1**  *Quantum processes specified by $\kappa$ are immanental processes for the associated correlation kernel.*

A direct consequence of this basic result is for instance simplicity of determinantal processes.

**Examples**

**Example 1** Quantum Pólya processes. (cf. Bach and Zessin[?], (2017))
We consider the realm of early quantum mechanics: $X$ is an infinite discrete space and $\lambda$ a point measure on $X$ which is bounded in the sense that $\sup_y \lambda(y) < \infty$. The quantum interaction is the kernel

$$\kappa(x, y) = 1_{\{0\}}(x - y), \quad x, y \in X.$$ 

Its cluster measure is

$$L \varphi = \sum_{m=1}^{\infty} \varepsilon^{m-1} z^m \sum_{x \in X} \varphi(m\delta_x) \lambda(x)^m.$$ 

The corresponding point process in $X$ exists for $z < 1$ small enough and is called the Pólya process. Its correlation kernel is

$$K_\kappa^{\varepsilon} = \frac{z}{1 - \varepsilon z} \kappa;$$ 

and its correlation function is

$$(x_1, \ldots, x_k) \mapsto \left(\frac{z}{1 - \varepsilon z}\right)^k \varphi_\kappa^{\varepsilon}(\delta x_1 + \cdots + \delta x_k).$$
The particle number $\zeta_B$ in a finite subset $B$ of $X$ has the distribution

$$P(\zeta_B = k) = (1 - \varepsilon z)^{\lambda(B)z^k} \frac{\lambda(B)^{\varepsilon[k]}}{k!},$$

where $a^{\varepsilon[k]} = a(a+1\varepsilon)(a+2\varepsilon)\cdots(a+(k-1)\varepsilon)$. If $\varepsilon = +1$ this law is negative binomial and otherwise binomial for the parameters $(\lambda(B), \frac{1-z}{z})$.

**Example 2** Ideal Bosons and Fermions. (cf. Ginibre[?], JMP 6 (1965))

$X = \mathbb{R}^d$ with Lebesgue’s measure $\lambda$. Ginibre isolated the quantum interaction

$$\kappa(x, y) = \frac{1}{w^d} \exp \left(-\frac{|x - y|^2}{2\beta}\right), \quad x, y \in X.$$
\( \beta \) is the inverse temperature, \( w = \sqrt{2\pi\beta} \) the thermal wave length. For \( z < 1 \) the associated point process exists and is called the ideal Bose gas if \( \varepsilon = +1 \), otherwise the ideal Fermi gas.

**Example 3** Quantum renewal processes. (cf. Macchi [?])

\( X = \mathbb{R} \) with Lebesgue measure \( \lambda \). Macchi considered the quantum interaction

\[
\kappa(x, y) = \exp\left(-\frac{|x - y|}{\alpha}\right), \quad \alpha > 0.
\]

The associated process is a Quantum renewal process. Again these are immanantal processes with a correlation kernel having the same form as \( \kappa \).

**Example 4** Quantum Ginibre processes. (Ginibre[?], JMP 6 (1965))
$X = \mathbb{C}$ with $\lambda(dx) = \exp(-|x|^2)\frac{1}{\pi} \, dx$ the standard Gaussian measure on $\mathbb{C}$. Ginibre considered the $\mathbb{C}$–valued exponential kernel

$$\kappa(x, y) = \exp x\bar{y}, \quad x, y \in \mathbb{C}.$$ 

Note that all other examples until now are real-valued and thereby also classical pair interactions. The associated process $\mathcal{G}$ built on the exponential kernel exists for all $0 < z < 1$ and is an immanantal process with correlation kernel

$$K = \frac{z}{1 - \varepsilon z} \kappa.$$ 

$\mathcal{G}$ is the Quantum Ginibre process. Its intensity measure is

$$\nu^{\mathcal{G}}_1(dx) = \frac{z}{1 - \varepsilon z} \frac{1}{\pi} \, dx.$$ 

Some implications of Quantum interactions
Consider again a Quantum process $P = \mathcal{G}_L$ in $X = \mathbb{R}^d$ with Lebesgue’s measure $\lambda$. By Proposition 1 we know already that $P$ is an immanantal process. Thus we know in principle all correlation functions.

$P$ is invariant under Euclidean motions, we then say that $P$ is stationary, if $L$ has this property. In all our examples one can show that $L$ is stationary so that the associated Quantum process has this property too. Moreover, in all examples the Quantum process is mixing, in particular ergodic. This implies an ergodic behaviour for a large class of additive functionals of the process; in particular laws of large numbers for the particle density of quantum processes. Finally, all examples are mixing in the sense of Brillinger, so that even central limit theorems for a large class of additive functionals hold true.
A problem. In the case of classical systems the immanant is replaced by the Boltzmann factor $\exp -E_\phi(\delta x_1 + \cdots + \delta x_k)$ for a given classical potential $\phi$. The first three quantum interactions from above are also classical stable potentials (by an observation in Ruelle’s book); and one can construct for them Gibbs processes by another procedure than we used to obtain the Quantum processes.

Question. What is the difference between a Quantum and a Gibbs process for a given interaction?

Modification. For the exponential interaction the modulus is a classical stable potential. What is the difference between the Quantum process for the exponential interaction and the Gibbs process for its modulus?
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References


