Quantum-classical point processes.

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General situation

$(X, \mathcal{B}, \mathcal{B}_0)$ is a general phase space. $\mathfrak{X} = \mathcal{M}^{\cdot\cdot}_f(X)$ denotes the collection of configurations in X, i.e. the finite point measures in X.

The point of departure is a complex measure L on $\mathfrak{X}$ satisfying $\mathsf{L}\{o\} = 0$. It is the generator of the main object

$$\Im_{\mathsf{L}} = \exp\left(\mathsf{L} - \mathsf{L}(1)\Delta_o\right).$$

Here the right hand side is defined as $\exp \mathsf{K} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{K}^{*n}$. $\Im_{\mathsf{L}}$ is a complex measure which is normalized in the sense $\Im_{\mathsf{L}}\,\mathbf{1} = 1$, and has the representation

$$\Im_{\mathsf{L}} = \frac{1}{\Xi} \exp \mathsf{L}, \qquad \Xi = \exp \mathsf{L}\,\mathbf{1}. \tag{1}$$

We are only interested in the case where $\Im_{\mathsf{L}}$ is a law, i.e. a probability P on $\mathfrak{X}$, i.e. a finite point process in X.

In typical situations a canonical procedure allows to extend these laws to processes with infinitely many particles.

Interpretation of $\Im_{\mathsf{L}}$ According to (1) $\Im_{\mathsf{L}}$ then realizes finite configurations of particles in X which are the result of finitely many independent superpositions of finite clusters (which are generated by L).

A special class of L**: Quantum processes**

We are given a *quantum interaction* κ, i.e. a Hermitean kernel κ of positive type (=positive definit) together with $z > 0$ and

$\varepsilon \in \{-1, +1\}$. We then consider the complex pseudomeasure

$$\mathsf{L}\ \varphi = \sum_{m=1}^{\infty} \varepsilon^{m-1} \frac{z^m}{m} \int_{X^m} \varphi(\delta_{x_1} + \cdots + \delta_{x_m}) \kappa(x_1, x_2) \kappa(x_2, x_3) \cdots$$

$$\kappa(x_m, x_1)\ \mathsf{d}\, x_1 \ldots \mathsf{d}\, x_m. \tag{2}$$

Here and in the sequel $\lambda(\mathsf{d}\, x) = \mathsf{d}\, x$ denotes a locally finite reference measure in the phase space. Under general and natural additional integrability assumptions on κ one then can show for z small enough the existence of an infinitely extended point process $P = \Im_{\mathsf{L}}$ in X, which is locally specified by the measures

$$\varrho_k(\mathsf{d}\, x_1 \ldots \mathsf{d}\, x_k) = z^k \wp_\varepsilon^\kappa(\delta_{x_1} + \cdots + \delta_{x_k})\ \mathsf{d}\, x_1 \ldots \mathsf{d}\, x_k. \tag{3}$$

The density $\wp_\varepsilon^\kappa$ is the *immanant of* (κ, ε); i.e. in the case $\varepsilon = -1$ the determinant $\det \Big(\kappa(x_i, x_j)\Big)_{i,j=1}^{k}$ and if $\varepsilon = +1$ the permanent of this matrix. This process $P = \Im_{\mathsf{L}}$ we call *quantum process specified by* (κ, ε).

One can show that L is of first order, and thereby P satisfies the socalled *cluster equation*

$$\mathcal{C}_P = \mathcal{C}_{\mathsf{L}} \star P, \tag{4}$$

which in some sense replaces the DLR-equation in the case of classical systems. Here $\mathcal{C}_P$ denotes the *Campbell measure of* P which contains the information of all moment measures of P. The operation $\star$ is some version of a convolution.

The cluster equation immediately implies that the intensity of P is

$$\nu_P^1(\mathrm{d}\,x) = \nu_{\mathsf{L}}^1(\mathrm{d}\,x) = K_\kappa^\varepsilon(x,x).\,\mathrm{d}\,x$$

Here

$$K_\kappa^\varepsilon = \sum_{m=1}^{\infty} \varepsilon^{m-1} z^m \kappa^{*m}$$

is the socalled *correlation kernel for* κ.

Consider the factorial moment measures $\check{\nu}_P^k$ of order $k \geq 2$ of P. They are defined by

$$\check{\nu}_P^k f = \int_{\mathcal{M}^{\cdot\cdot}(X)} \check{\mu}^k(f)\, P(\mathrm{d}\,\mu), \qquad f \in \mathcal{K}(X).$$

Here

$$\check{\mu}^k(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k) = \mu(\mathrm{d}\,x_1)\,(\mu - \delta_{x_1})\,(\mathrm{d}\,x_2) \cdots \left(\mu - \sum_{j=1}^{k-1} \delta_{x_j}\right)(\mathrm{d}\,x_k).$$

One can show that they are dominated by λ^k:

$$\check{\nu}_P^k(\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k) = \wp_\varepsilon^{K_\kappa^\varepsilon}(\delta_{x_1} + \cdots + \delta_{x_k}).\,\mathrm{d}\,x_1 \ldots \mathrm{d}\,x_k. \tag{5}$$

The immanant appearing as a density is the correlation function of P. Equation (5) signifies that P is an immanantal process for

K_κ^ε, thus a permanental process if $\varepsilon = +1$ and a determinantal process if $\varepsilon = -1$.

To summarize:

Proposition 1 *Quantum processes specified by κ are immanantal processes for the associated correlation kernel.*

A direct consequence of this basic result is for instance simplicity of determinantal processes.

Examples

Example 1 `Quantum Pólya processes. (cf. Bach and Zessin[?], (2017))` *We consider the realm of early quantum mechanics:* X *is an infinite discrete space and* λ *a point measure on* X *which is bounded in the sense that* $\sup_y \lambda(y) < \infty$. *The quantum interaction is the kernel*

$$\kappa(x,y) = \mathbf{1}_{\{0\}}(x-y), \qquad x,y \in X.$$

Its cluster measure is

$$\mathsf{L}\,\varphi = \sum_{m=1}^{\infty} \varepsilon^{m-1}\frac{z^m}{m}\sum_{x\in X}\varphi(m\delta_x)\lambda(x)^m.$$

The corresponding point process in X *exists for* $z < 1$ *small enough and is called the Pólya process. Its correlation kernel is*

$$K_\kappa^\varepsilon = \frac{z}{1-\varepsilon z}\kappa;$$

and its correlation function is

$$(x_1,\ldots,x_k) \longmapsto \left(\frac{z}{1-\varepsilon z}\right)^k \wp_\varepsilon^\kappa(\delta_{x_1}+\cdots+\delta_{x_k}).$$

The particle number ζ_B in a finite subset B of X has the distribution

$$P(\zeta_B = k) = (1 - \varepsilon z)^{\lambda(B)} \frac{z^k}{k!} \lambda(B)^{\varepsilon[k]},$$

where $a^{\varepsilon[k]} = a(a+1\varepsilon)(a+2\varepsilon)\cdots(a+(k-1)\varepsilon)$. *If* $\varepsilon = +1$ *this law is negative binomial and otherwise binomial for the parameters* $(\lambda(B), \frac{1-z}{z})$.

Example 2 `Ideal Bosons and Fermions. (cf. Ginibre[?], JMP 6 (1965))`

$X = \mathbb{R}^d$ *with Lebesgue's measure* λ. *Ginibre isolated the quantum interaction*

$$\kappa(x, y) = \frac{1}{\mathrm{w}^d} \exp\left(-\frac{|x-y|^2}{2\beta}\right), \qquad x, y \in X.$$

β *is the inverse temperature,* $\mathsf{w} = \sqrt{2\pi\beta}$ *the thermal wave length. For* $z < 1$ *the associated point process exists and is called the ideal Bose gas if* $\varepsilon = +1$*, otherwise the ideal Fermi gas.*

Example 3 `Quantum renewal processes. (cf. Macchi [?])`

$X = \mathbb{R}$ *with Lebesgue measure* λ*. Macchi considered the quantum interaction*

$$\kappa(x,y) = \exp\left(-\frac{|x-y|}{\alpha}\right), \qquad \alpha > 0.$$

The associated process is a Quantum renewal process. Again these are immanantal processes with a correlation kernel having the same form as κ*.*

Example 4 `Quantum Ginibre processes. (Ginibre[?], JMP 6 (1965))`

$X = \mathbb{C}$ *with* $\lambda(\mathrm{d}\, x) = \exp(-|x|^2)\frac{1}{\pi}\,\mathrm{d}\, x$ *the standard Gaussian measure on* $\mathbb{C}$*. Ginibre considered the* $\mathbb{C}-$*valued exponential kernel*

$$\kappa(x, y) = \exp\, x\bar{y}, \qquad x, y \in \mathbb{C}.$$

Note that all other examples until now are real-valued and thereby also classical pair interactions. The associated process $\mathfrak{G}$ *built on the exponential kernel exists for all* $0 < z < 1$ *and is an immanantal process with correlation kernel*

$$K = \frac{z}{1 - \varepsilon z}\kappa.$$

$\mathfrak{G}$ *is the Quantum Ginibre process. Its intensity measure is* $\nu^1_{\mathfrak{G}}(\mathrm{d}\, x) = \frac{z}{1-\varepsilon z}\frac{1}{\pi}\,\mathrm{d}\, x$*.*

Some implications of Quantum interactions

Consider again a Quantum process $P = \Im_{\mathsf{L}}$ in $X = \mathbb{R}^d$ with Lebuesgue's measure λ. By Proposition 1 we know alredy that P is an immanantal process. Thus we know in principle all correlation functions.

P is invariant under Euclidean motions, we then say that P is stationary, if L has this property. In all our examples one can show that L is stationary so that the associated Quantum process has this property too. Moreover, in all examples the Quantum process is mixing, in particular ergodic. This implies an ergodic behaviour for a large class of additive functionals of the process; in particular laws of large numbers for the particle density of quantum processes. Finally, all examples are mixing in the sense of Brillinger, so that even central limit theorems for a large class of additive functionals hold true.

`A problem.` In the case of classical systems the immanant is replaced by the Boltzmann factor exp $-E_\phi(\delta_{x_1}+\cdots+\delta_{x_k})$ for a given classical potential ϕ. The first three quantum interactions from above are also classical stable potentials (by an observation in Ruelle's book); and one can construct for them Gibbs processes by another procedure than we used to obtain the Quantum processes.

`Question.` What is the difference between a Quantum and a Gibbs process for a given interaction?

`Modification.` For the exponential interaction the modulus is a classical stable potential. What is the difference between the Quantum process for the exponential interaction and the Gibbs process for its modulus?

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