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Construction of Dynamical Semigroups by Regularisation à la Kato

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Résumé

1.Dynamical Semigroups
2.Regularisation Theorem
3.Applications

This talk is based on a joint work with A.F.M.ter Elst (Auckland)

1. Dynamical Semigroups

• Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$ and $\mathcal{L}(\mathcal{H})$ be the Banach space of bounded operators with the subspace $\mathfrak{C}_1 = \mathfrak{C}_1(\mathcal{H})$ of all trace-class operators. A bounded operator is positive $u \geq 0$, if $(u\,x, x)_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$.

• Let $\mathfrak{C}_1^+ = \{u \in \mathfrak{C}_1 : u \geq 0\}$. Then $\mathfrak{C}_1^+$ is a closed cone with the trace-norm $\|u\|_{\mathfrak{C}_1} = \mathrm{Tr}_{\mathcal{H}}(u)$, for all $u \in \mathfrak{C}_1^+$.

• Let $\mathfrak{C}_1^{\mathrm{sa}}$ be the Banach space over $\mathbb{R}$ of all self-adjoint operators of $\mathfrak{C}_1$, with generating positive cone $\mathfrak{C}_1^+ \subset \mathfrak{C}_1^{\mathrm{sa}}$ on which the trace-norm is additive: $\|u + v\|_{\mathfrak{C}_1} = \|u\|_{\mathfrak{C}_1} + \|v\|_{\mathfrak{C}_1}$.

• Operator $A : \mathfrak{D}(A) \to \mathfrak{C}_1^{\mathrm{sa}}$ with *domain* : $\mathfrak{D}(A) \subset \mathfrak{C}_1^{\mathrm{sa}}$, is called **positivity preserving** if $Au \geq 0$ for all $u \in \mathfrak{D}(A)^+ := \{u \in \mathfrak{D}(A) : u \geq 0\}$. A *semigroup* $\{S_t\}_{t\geq 0}$ on $\mathfrak{C}_1^{\mathrm{sa}}$ is called **positivity preserving** if the map $S_t : \mathfrak{C}_1^+ \to \mathfrak{C}_1^+$, for all $t \geq 0$.

• Let H be generator of the *positivity preserving* and **contraction** C_0-semigroup $\{e^{-tH}\}_{t\geq 0}$ on $\mathfrak{C}_1^{\mathrm{sa}}$ (*Dynamical Semigroup*). Let $K : \mathfrak{D}(H) \to \mathfrak{C}_1^{\mathrm{sa}}$ be a **positivity preserving** operator such that

$$\mathrm{Tr}_{\mathcal{H}}(Ku) \leq \mathrm{Tr}_{\mathcal{H}}(Hu) \ , \ \forall u \in \mathfrak{D}(H)^+ .$$

So, if the operator H is *positivity preserving*, then the operator K is H-bounded, but with the **relative bound** equals to **one**.

• **Q:** Whether operator $(H - K)$, or its closed extension, is still generator of a C_0-semigroup $\{T_t\}_{t\geq 0}$?

• **Kato (1954)** solved this perturbation problem when the operator H is a *positivity preserving* map. To this end he proposed a **regularisation** of the perturbation K by the one-parametric family $\{rK\}_{r\in[0,1)}$ and by taking finally the limit $r\uparrow 1$.

• Our aim was *twofold*: to consider a more general (*functional*) regularisation à la Kato and to *relax* the condition that the **operator** H is positivity preserving to the condition that H is **generator** of a positivity preserving semigroup. The last is indispensable for construction of the *Quantum Dynamical Semigroups*.

• In the latter case, according the Kossakowski–Lindblad–Davies *Ansatz*, these semigroups must be *completely positive* and *trace-preserving* maps: $\mathrm{Tr}_{\mathcal{H}}(T_t w) = \mathrm{Tr}_{\mathcal{H}}(w)$, $w \in \mathfrak{C}_1^{\mathrm{sa}}$.

2. Regularisation Theorem

• **Definition:** Let $(K_\alpha)_{\alpha\in J}$ be a **net** such that $K_\alpha\colon \mathfrak{D}(H) \to \mathfrak{C}_1^{\text{sa}}$ for all $\alpha \in J$. We call the family $\{K_\alpha\}_{\alpha\in J}$ a **regularisation** of the operator K if the following *four* conditions are valid:

• K_α is positivity preserving for all $\alpha \in J$.

• For all $\alpha \in J$ there exist $a_\alpha \in [0, \infty)$ and $b_\alpha \in [0, 1)$ such that

$$\text{Tr}_{\mathcal{H}}(K_\alpha u) \leq a_\alpha \,\text{Tr}_{\mathcal{H}}(u) + b_\alpha \,\text{Tr}_{\mathcal{H}}(Hu)$$

for all $u \in \mathfrak{D}(H)^+$.

• $K_\alpha \leq K_\beta \leq K$ for all $\alpha, \beta \in J$ with $\alpha \leq \beta$.

• $\lim_\alpha((K_\alpha u)x, x)_{\mathcal{H}} = ((Ku)x, x)_{\mathcal{H}}$ for all $u \in \mathfrak{D}(H)^+$ and $x \in \mathcal{H}$.

• Let $J = [0, 1)$ and $K_r = rK$ for $r \in J$ ($a_\alpha = 0$, $b_\alpha = r$). If $\text{Tr}_{\mathcal{H}}(Ku) \leq \text{Tr}_{\mathcal{H}}(Hu)$, $\forall u \in \mathfrak{D}(H)^+$, and *if* H is positivity preserving, then $\{K_r\}_{r\in J}$ is the one-parameter **Kato regularisation** of the operator K.

• **Theorem 1:** Let H be the generator of a **positivity preserving contraction** C_0-semigroup on $\mathfrak{C}_1^{\mathrm{sa}}$. Let $K\colon \mathfrak{D}(H) \to \mathfrak{C}_1^{\mathrm{sa}}$ be a positivity preserving operator and suppose that

$$\mathrm{Tr}_{\mathcal{H}}(Ku) \leq \mathrm{Tr}_{\mathcal{H}}(Hu)$$

for all $u \in \mathfrak{D}(H)^+$. Let $\{K_\alpha\}_{\alpha\in J}$ be a **regularisation** of K. Set $L_\alpha = H - K_\alpha$ for all $\alpha \in J$. Then:

(a) For all $\alpha \in J$ the operator L_α is the generator of a positivity preserving contraction C_0-semigroup $\{T_t^\alpha := e^{-tL_\alpha}\}_{t\geq 0}$ on $\mathfrak{C}_1^{\mathrm{sa}}$.

(b) If $t > 0$, then $\lim_\alpha T_t^\alpha u$ exists on $\mathfrak{C}_1^{\mathrm{sa}}$ for all $u \in \mathfrak{C}_1^{\mathrm{sa}}$.

For all $t > 0$ we define $T_t\colon \mathfrak{C}_1^{\mathrm{sa}} \to \mathfrak{C}_1^{\mathrm{sa}}$ by $T_t u = \lim_\alpha T_t^\alpha u$.

(c) $\{T_t := e^{-tL}\}_{t>0}$ is a positivity preserving contraction C_0-semigroup on $\mathfrak{C}_1^{\mathrm{sa}}$ for which the generator $L = (H-K)^\sim$ is **a** closed extension of the operator $(H-K)$, $\mathrm{dom}(H-K) = \mathrm{dom}(H)$.

• **Theorem 2:** Let L' be another closed extension of the operator $(H-K)$, $\mathrm{dom}(H-K) = \mathrm{dom}(H)$, such that L' generates a C_0-semigroup $\{T'_t\}_{t\geq 0}$. Then $T'_t \geq T_t$ for all $t > 0$.

• **Remark:** Similarly to the Kato one-parameter r-regularisation, the semigroup $\{T_t\}_{t\geq 0}$ constructed in Theorem 1 by the *functional regularisation* $\{K_\alpha\}_{\alpha\in J}$ is called **minimal**.

• **Theorem 3:** If in addition to conditions of Theorem 1, one supposes that

$$\mathrm{Tr}_{\mathcal{H}}(Hu - Ku) = 0 \ , \ \forall u \in \mathrm{dom}(H) \ ,$$

and that $\mathfrak{D}(H)$ is a *core* for the generator L. Then the positivity preserving contraction C_0-semigroup $\{T_t = e^{-tL}\}_{t\geq 0}$ is **trace-preserving**: $\mathrm{Tr}_{\mathcal{H}}(T_t w) = \mathrm{Tr}_{\mathcal{H}}(w)$, $w \in \mathfrak{C}_1^{\mathrm{sa}}$.

3.Application: Open Quantum Oscillator

• Let b and b^* be the boson annihilation and creation operators defined in the Fock space $\mathcal{H} = \mathfrak{F}$ generated by a cyclic vector Ω. The **isolated** system is a quantum oscillator:

$$h = E\, b^* b \ , \ \ E > 0 \ .$$

• **Open system à la Kossakowski–Lindblad–Davies**

Formal non-Hamiltonian evolution of density matrix $\rho(t) \in \mathfrak{C}_1^{\mathsf{sa}}$:

$$\partial_t \rho(t) = -L\rho(t) \ , \ \ L = H - K \ ,$$

$$H\rho = i\,[h, \rho] + \frac{1}{2}\Big[(\sigma_-\, b^*\, b + \sigma_+\, b\, b^*)\rho + \rho\,(\sigma_-\, b^*\, b + \sigma_+\, b\, b^*)\Big],$$

$$K\,\rho = \sigma_-\, b\,\rho\, b^* + \sigma_+\, b^*\,\rho\, b \ , \ \ \text{pumping – leaking rates} : \ \sigma_\pm \geq 0 \ .$$

- **Photon-number cut-off regilarisation**

1. Since in $L := H - K$ the operator K has **relative bound one** with respect to H, we consider a **regularisation** generated by the family of *projections* $\{P_N\}_{N\in\mathbb{N}}$, where for all $N \in \mathbb{N}$ the projection $P_N : \mathfrak{F} \to \mathfrak{F}_N$.

2. The number of bosons in the subspace $\mathfrak{F}_N$ is **bounded**: for $\psi \in \mathfrak{F}$ one has $b^*b(P_N\psi) \leq N\|\psi\|^2_{\mathfrak{F}}$. $\mathfrak{F}_N \subset \mathfrak{F}_{N+k}$ for all $k \in \mathbb{N}$ and $\lim_{N\to\infty} P_N\psi = \psi$ for all $\psi \in \mathfrak{F}$ verifying the conditions of regularisation in **Definition**.

3. For all $N \in \mathbb{N}$ define the **particle number cut-off** regularisation $K_N \in \mathcal{L}(\mathfrak{C}_1^{\text{sa}})$ of the operator K by

$$K_N\rho := \sigma_- (b^* P_N)^* \rho (b^* P_N) + \sigma_+ (b P_N)^* \rho (b P_N)$$

- **Theorem 4:**

If the parameters $\sigma_\pm$ satisfy the condition $\sigma_+ < \sigma_-$, then:
(i) Domain dom(H) is a *core* for the generator
$M = \lim_{N\to\infty}(H - K_N)$ of a semigroup $\{T_t\}_{t\geq 0}$.
(ii) M is a closed extension of L.
(iii) The semigroup $\{T_t\}_{t\geq 0}$ is minimal and trace-preserving.

THANK YOU !