Dimerisation in quantum spin chains

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Joint work with **Bruno Nachtergaele**

Quantum spin chain

Chain $\{-\ell+1, -\ell+2, \dots, \ell\}$
Hilbert space $\mathcal{H}_\ell = \otimes_{x=-\ell+1}^{\ell} \mathbb{C}^{2S+1}$, with $S \in \frac{1}{2}\mathbb{N}$
Interactions given by projectors $P_{x,y}^{(0)}$ onto spin singlet:

$$P_{x,y}^{(0)} = \frac{1}{2S+1} \sum_{a,b=-S}^{S} (-1)^{a-b} |a, -a\rangle\langle b, -b|$$

Hamiltonian $H_\ell = -\sum_{x=-\ell+1}^{\ell-1} P_{x,x+1}^{(0)}$
The projector onto the spin singlet can be written in terms of the usual spin operators S_x^i:

$$S = \tfrac{1}{2} : P_{x,y}^{(0)} = -\vec{S}_x \cdot \vec{S}_y + \tfrac{1}{4} \qquad \text{Heisenberg AF}$$

$$S = 1 : P_{x,y}^{(0)} = \tfrac{1}{2}(\vec{S}_x \cdot \vec{S}_y)^2 - \tfrac{1}{3}$$

Dimerisation

Gibbs state:

$$\langle a\rangle_{\beta,\ell} = \frac{1}{\mathrm{Tr}\ \mathrm{e}^{-\beta H_\ell}} \mathrm{Tr}\, a\, \mathrm{e}^{-\beta H_\ell}\ ,$$

Ground state:

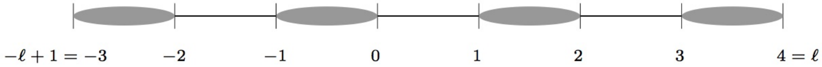
$$\langle\cdot\rangle_{\infty,\ell} = \lim_{\beta\to\infty} \langle\cdot\rangle_{\beta,\ell}$$

Theorem [Nachtergaele, U '16]

For $S > 31$, there exists $c(S) > 0$ such that

$$\langle P^{(0)}_{x,x+1}\rangle_{\infty,\ell} - \langle P^{(0)}_{x-1,x}\rangle_{\infty,\ell} > c(S),$$

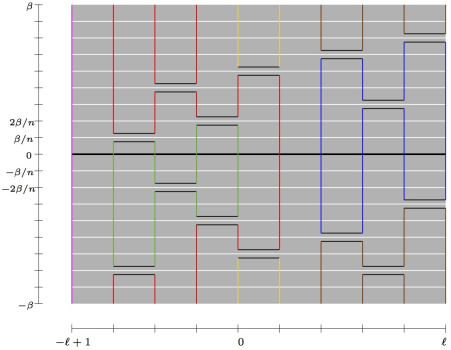
for all $x \in \{-\ell+3, -\ell+5, \dots, \ell-1\}$, uniformly in $\ell \in \mathbb{N}$



Method of proof

- Random loop representation of **[Aizenman-Nachtergaele '94]**
- Contours with respect to background of dimerised short loops
- Peierls argument

Random loop representation



Relevant probability measure: $\mu_{\beta,\ell,n}(\omega) = \frac{1}{Z_n(\beta,\ell)} \left(\frac{\beta}{n}\right)^{|\omega|} (2S+1)^{|\mathcal{L}(\omega)|-|\omega|}$

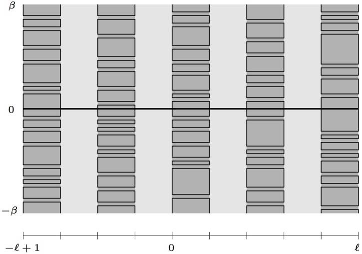
Proposition

(a) $\mathrm{Tr}\, \mathrm{e}^{-2\beta H_\ell} = \lim_{n\to\infty} Z_n(\beta,\ell)$

(b) $\langle P^{(0)}_{x,x+1} \rangle_{\beta,\ell} = \frac{1}{(2S+1)^2} + \left(1 - \frac{1}{(2S+1)^2}\right) \lim_{n\to\infty} \mathbb{P}_{\beta,\ell,n}(x \leftrightarrow x+1)$

Background loops

The measure $\mu_{\beta,\ell,n}$ is biased towards configurations with many loops. Optimal way:



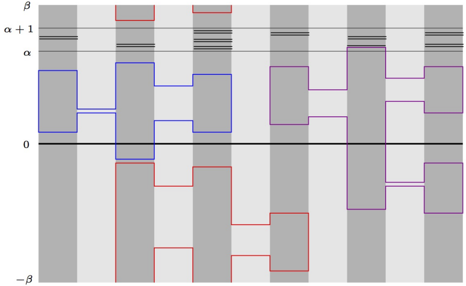
If these were typical configurations:

$$\mathbb{P}_{\beta,\ell,n}(x \leftrightarrow x+1) = 1 > 0 = \mathbb{P}_{\beta,\ell,n}(x \not\leftrightarrow x+1)$$

for all $x \in \{-\ell+1, -\ell+3, \dots, \ell-1\}$

Restricted set of configurations + contours

For integers $\alpha < \beta$, let $\Omega^{\alpha}_{\ell,n}$ denote the set of configurations:



Almost all configurations have this property for some $\alpha \in \mathbb{N}$:

Lemma

We have

$$\lim_{\beta\to\infty} \mathbb{P}_{\beta,\ell,n}\big(\cup_{\alpha=1}^{\beta} \Omega^{\alpha}_{\ell,n}\big) = 1$$

The limit $\beta \to \infty$ converges uniformly in n

Peierls argument

Event $E_x^{\circlearrowleft}$ where $(x,0)$ is surrounded by an external contour

Lemma

If $x \in \{-\ell+3, -\ell+5, \dots, \ell-1\}$, we have

$$\mathbb{P}_{\beta,\ell,n}(x \leftrightarrow x{+}1 | \Omega^\alpha_{\ell,n}) - \mathbb{P}_{\beta,\ell,n}(x{-}1 \leftrightarrow x | \Omega^\alpha_{\ell,n}) \geqslant 1 - 2\mathbb{P}_{\beta,\ell,n}(E_x^{\circlearrowleft} | \Omega^\alpha_{\ell,n})$$

Lemma

$$\mathbb{P}_{\beta,\ell,n}(E_x^{\circlearrowleft} | \Omega^\alpha_{\ell,n}) = \frac{1}{Z} \sum_{\omega \in E_x^{\circlearrowleft}} \big(\tfrac{\beta}{n}\big)^{|\omega|} \, (2S+1)^{|\mathcal{L}(\omega)|-|\omega|}$$

Peierls argument

Lemma

$$\mathbb{P}_{\beta,\ell,n}(E_x^{\circlearrowleft} | \Omega_{\ell,n}^{\alpha}) \leqslant \sum_{\gamma \in E_x^{\circlearrowleft} \cap \Omega_{D_\alpha}} \left(\tfrac{\beta}{n}\right)^{|\gamma|} (2S+1)^{-\frac{1}{2}|\gamma|} \left(1 + \tfrac{\beta}{n}\right)^{-\frac{1}{2}L(\gamma)}$$

One can write an explicit bound for the sum over all contours surrounding $(x, 0)$, and it is smaller than $\frac{1}{2}$ for S large enough

This proves dimerisation!

Peierls argument

Lemma

$$\mathbb{P}_{\beta,\ell,n}(E_x^{\circlearrowleft} | \Omega_{\ell,n}^{\alpha}) \leqslant \sum_{\gamma \in E_x^{\circlearrowleft} \cap \Omega_{D_\alpha}} \left(\tfrac{\beta}{n}\right)^{|\gamma|} (2S+1)^{-\frac12 |\gamma|} \left(1 + \tfrac{\beta}{n}\right)^{-\frac12 L(\gamma)}$$

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This proves dimerisation!