High temperature regime in spatial random permutations

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Definition: Nearest neighbor SRP with forced long cycle

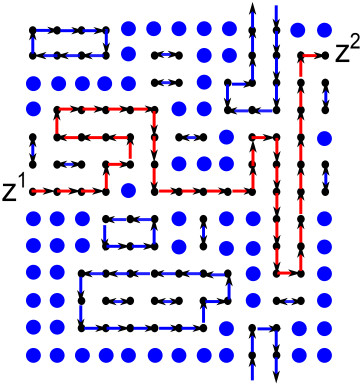
Set $\Lambda_L = [0, L]^d \cap \mathbb{Z}^d$ and fix two sites $z^1, z^2 \in \Lambda_L$

- **Forced open cycle between z^1 and z^2**

  $\mathcal{S}_{\Lambda_L}^{z^1 \to z^2}$ = set of bijections $\pi : \Lambda_L \setminus \{z^1\} \to \Lambda_L \setminus \{z^2\}$ with the constraint that $\pi(x) = x$ or $\pi(x) \sim x$ for all $x \in \Lambda_L$.

- **Gibbs measure** depending on $\alpha \in [0, \infty)$

$$\mathbb{P}_\Lambda(\{\pi\}) = \frac{1}{Z(\Lambda)} \exp\Big(-\alpha \sum_{x \in \Lambda} |\pi(x) - x|\Big),$$



- **Actual settings**

  1. *forced open cycle between opposite sides.*
     z_1 = centre of the boundary side of the box,
     $$\mathcal{S}_{\Lambda_L} = \sum_{z^2 \in \text{ opp. side to } z^1} \mathcal{S}_{\Lambda_L}^{z^1 \to z^2}$$

  2. *only cycles*, namely nearest-neighbours permutations $\pi : \Lambda \to \Lambda$.

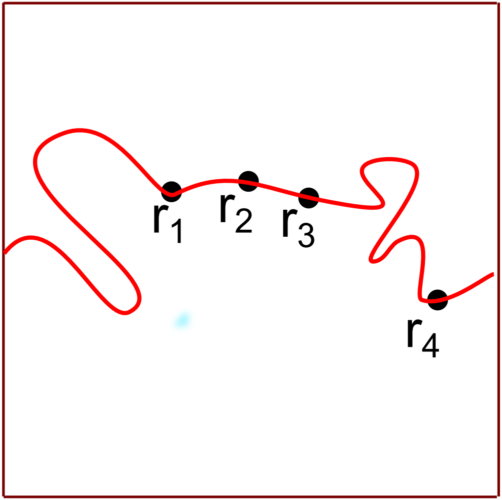
Difficulties and results

- **Difficulties and challenges**
  1. no classical spin system (bijections introduce constraints)
  2. energy of the open cycle much smaller than the energy of the system
  3. no comparison between the law of π and the law of π conditional on some local features.

- **Results** There exists $\alpha_c < \infty$ such that for all $\alpha > \alpha_c$,
  1. No long cycles exist
  2. Exponential decay of correlations
  3. Orstein-Zernike behaviour for the forced open cycle

in any dimension of $\mathbb{Z}^d$.

Orstein-Zernike method

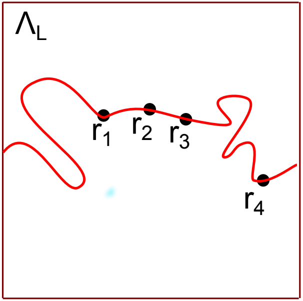


Orstein-Zernike method

Define a proper function on the sample space $f(\pi) = \{(r_1, \xi_1), (r_2, \xi_2), \ldots (r_N, \xi_N)\}$, such that

- Markov process $(r_i, \xi_i) \to (r_{i+1}, \xi_{i+1})$,

$$P\left(r_{i+1},\, \xi_{i+1} \;\middle|\; (r_1,\, \xi_1), \ldots (r_i, \xi_i)\right) = Q_L\left(r_{i+1}, \xi_{i+1}, r_i, \xi_i\right)$$



- Regeneration surfaces are defined to be symmetric under a reflection with respect to $r_{i,2}$. This implies that

$$E[(r_{i+1} - r_i) \cdot \mathbf{e}_2 \;\mid\; r_i, \xi_i] = 0.$$

  for any r_i, ξ_i.

- We prove that,

$$P(|r_{i+1} - r_i| > D\log(L) \;\mid\; r_i, \xi_i) \leqslant C\exp\{-\sqrt{D}\; c\}$$

  with C and c independent on ξ_i and positive.

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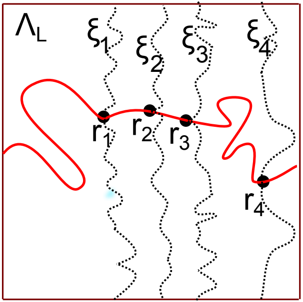

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Cycle growth process

We define a set of indep. r.v. $(\sigma_A)_{A \subset \mathbb{Z}^d}$, where σ_A is distributed like $\mathbb{P}_A$.

- **Step** 1:
  1. choose a site x_1 on the vertical line
  2. sample σ_Λ and keep only the cycle intersecting x_1, γ_1
  3. define $B_1 = \Lambda \setminus \gamma_1$
  4. define the "non matching set" $\mathcal{N}_1$
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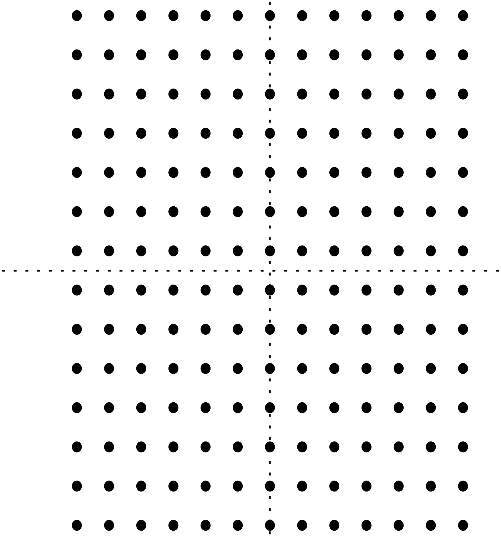

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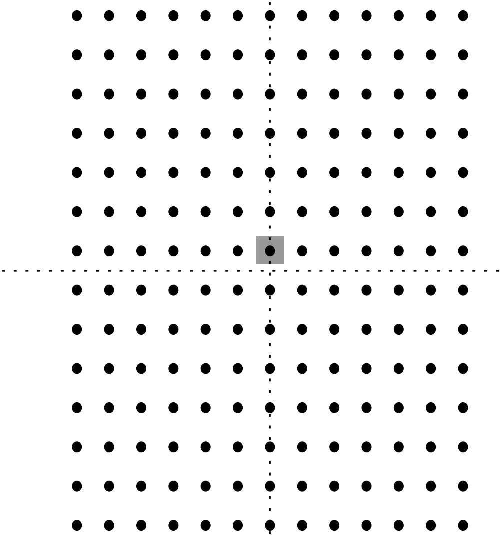

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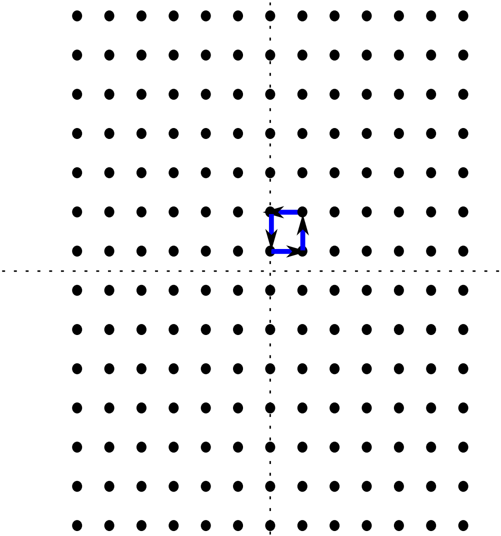

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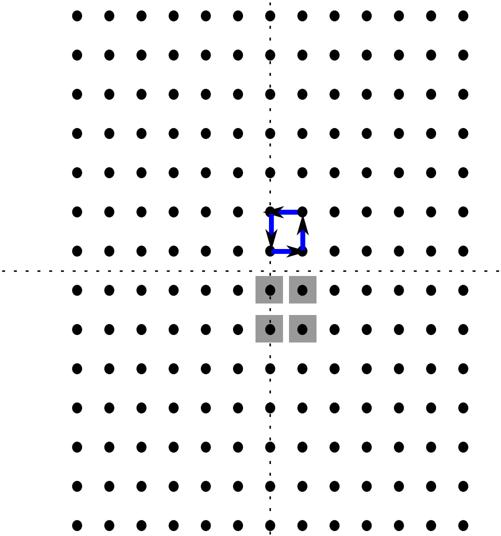

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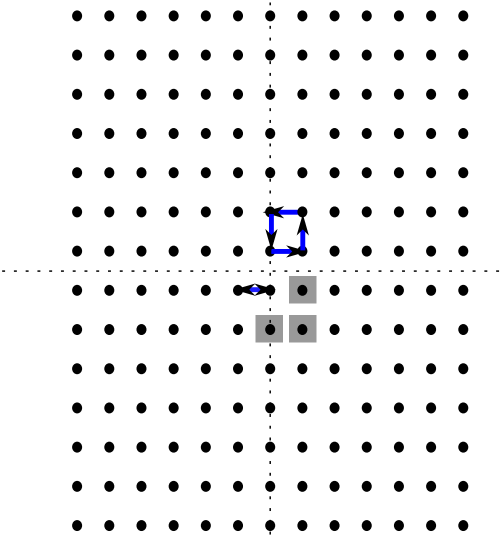

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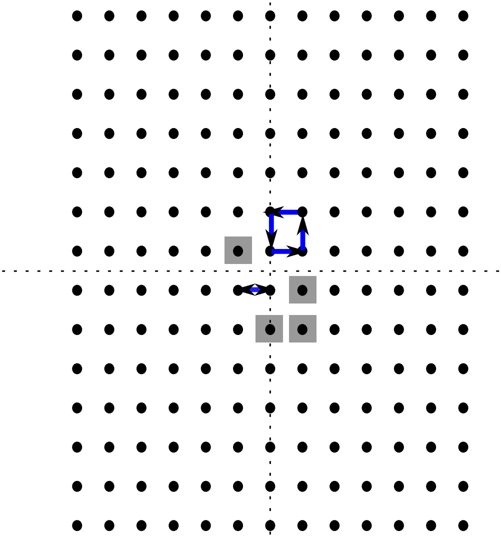

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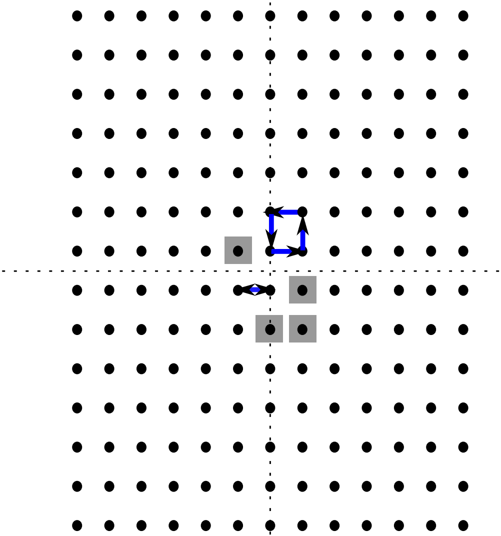

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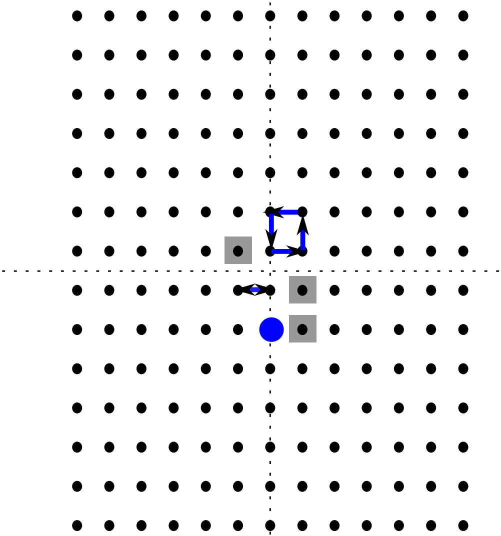

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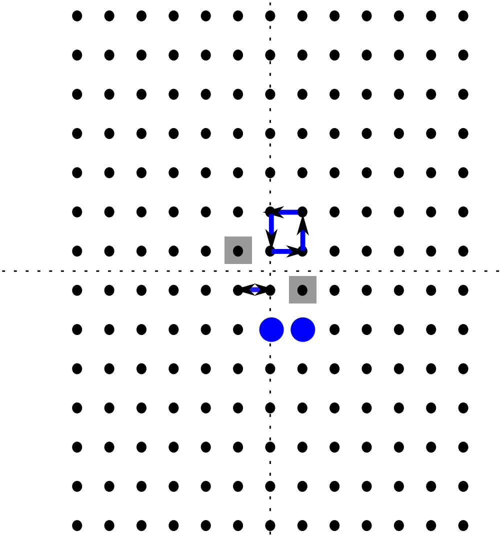

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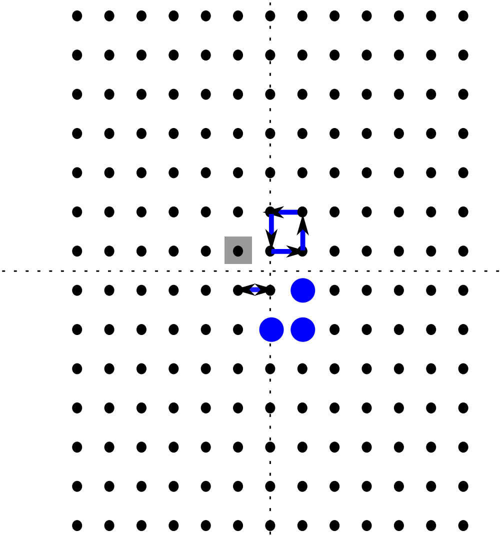

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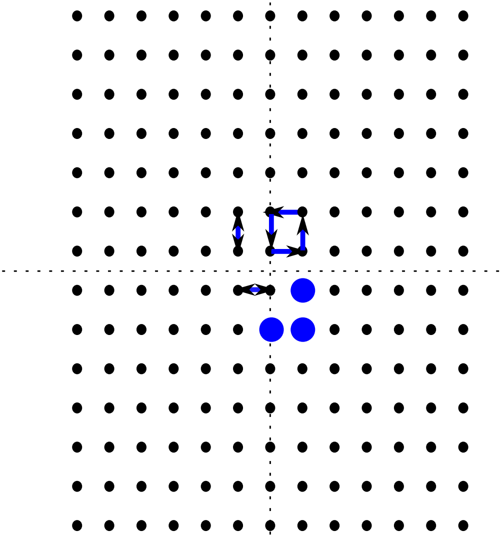

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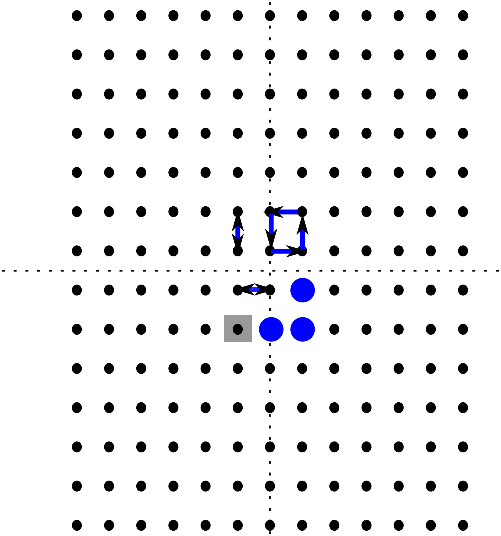

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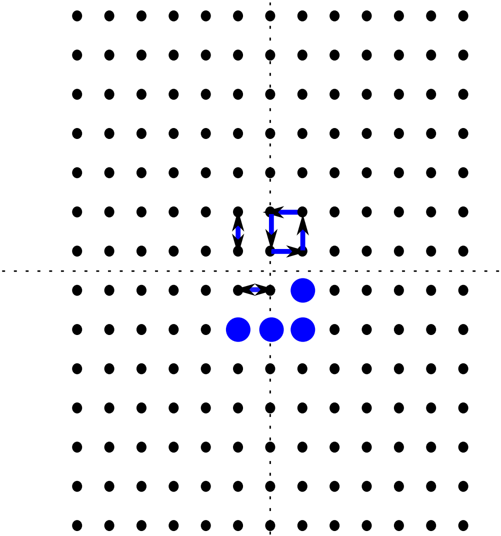

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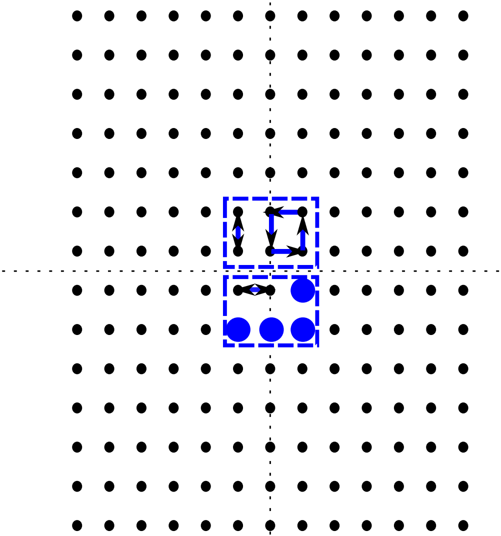
We have the first cluster. We start again from a new site on the vertical line.



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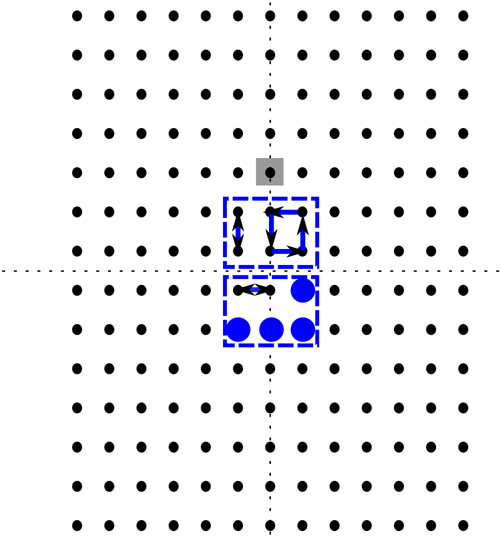

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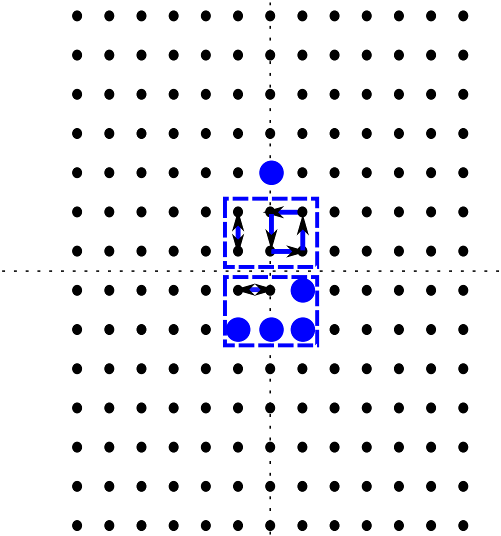

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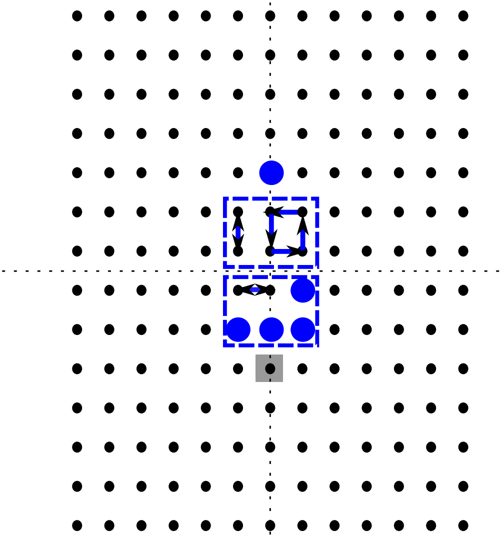

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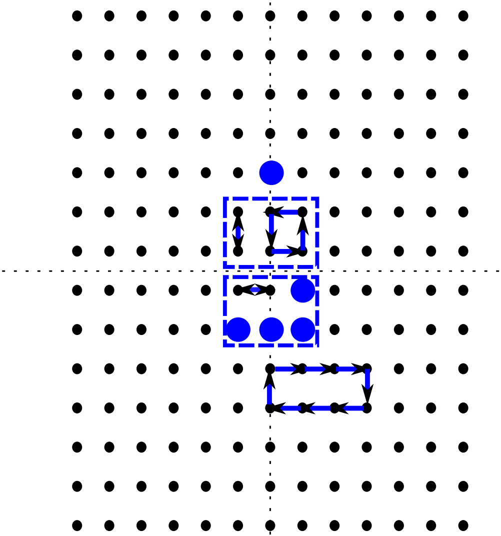

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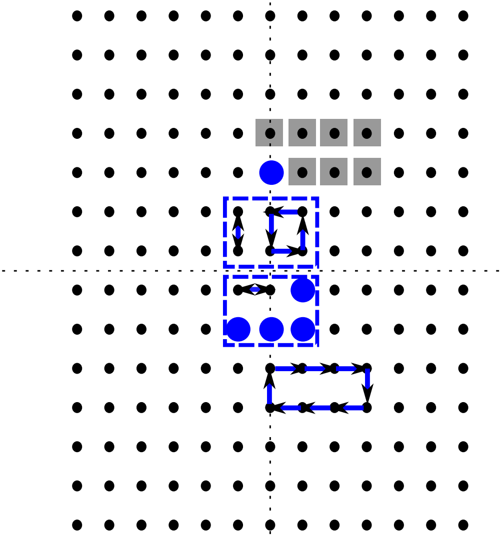

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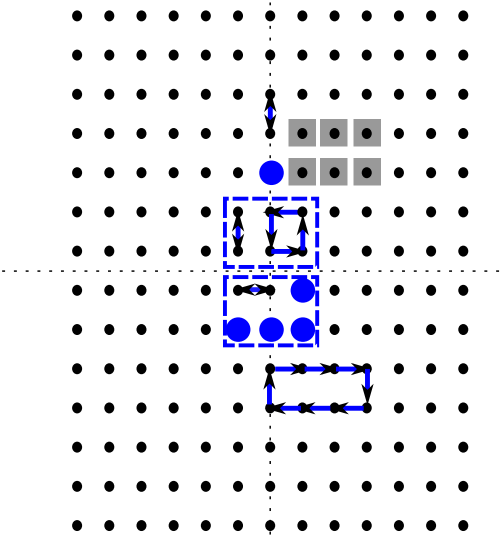

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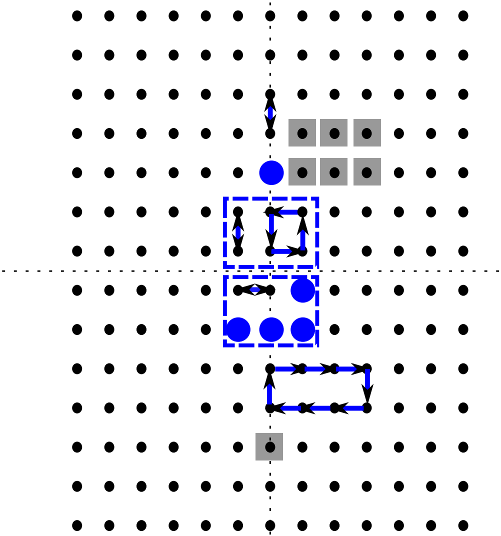

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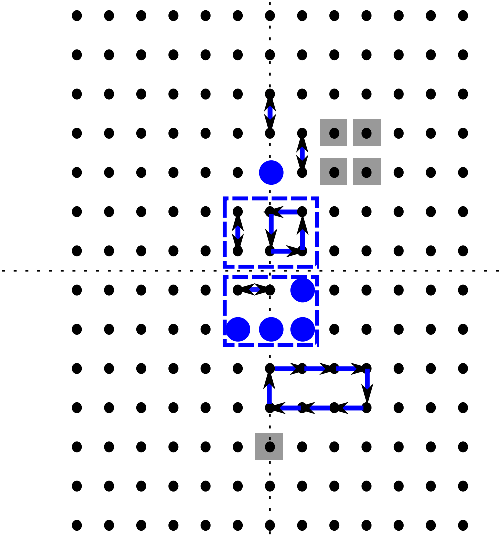

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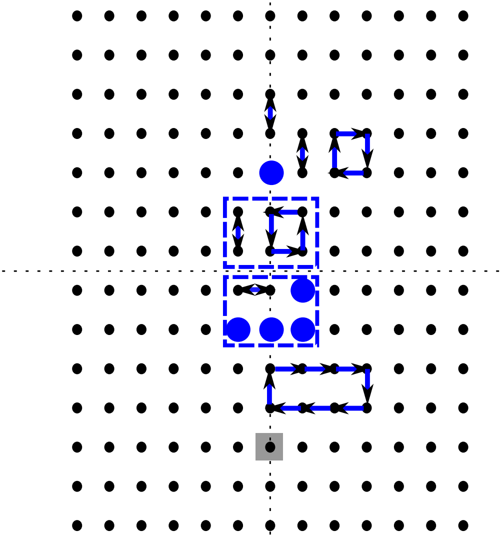

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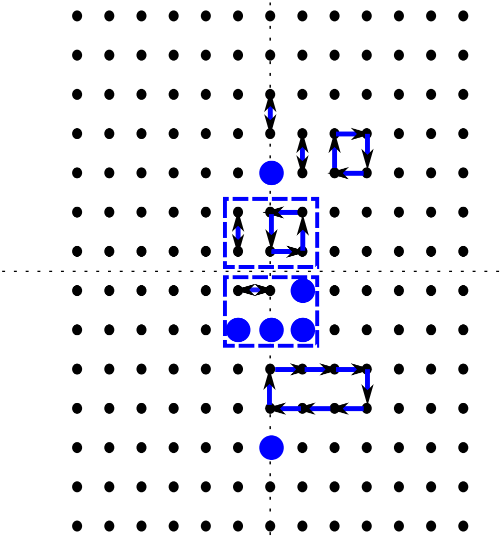

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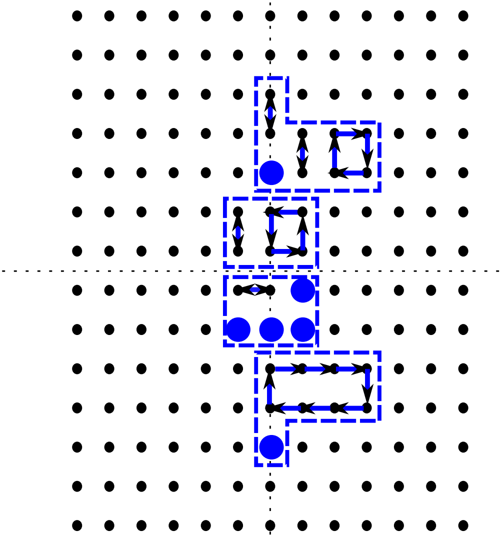

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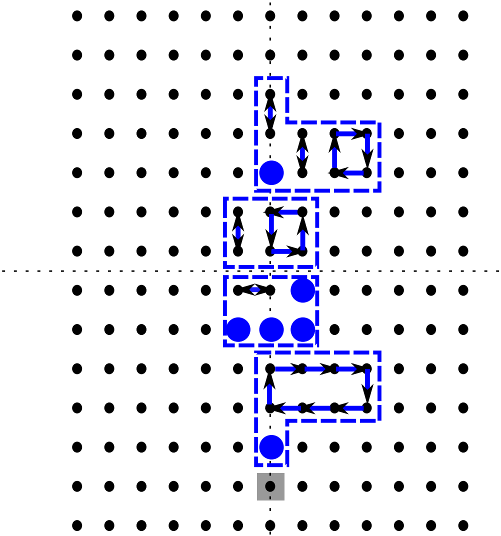

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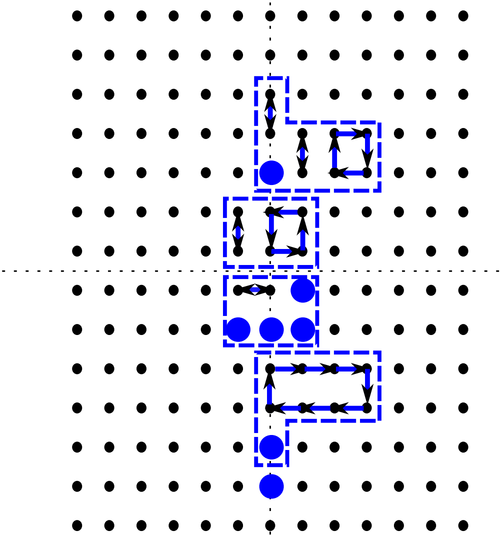

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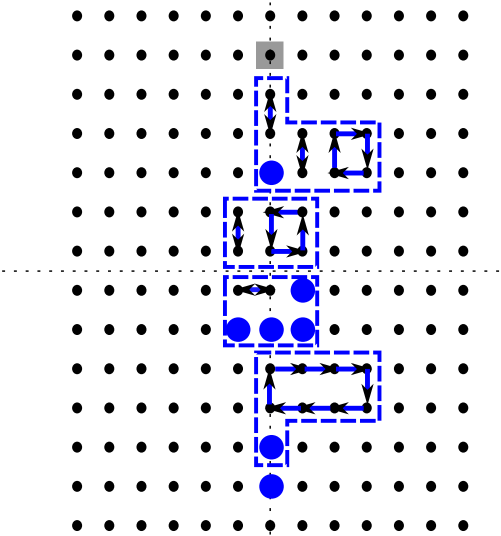

Cycle growth process

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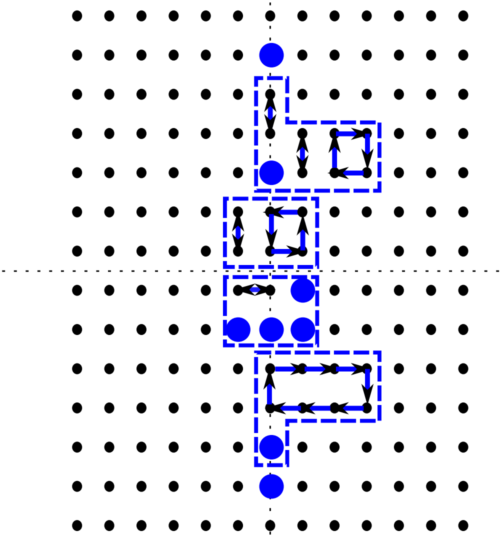

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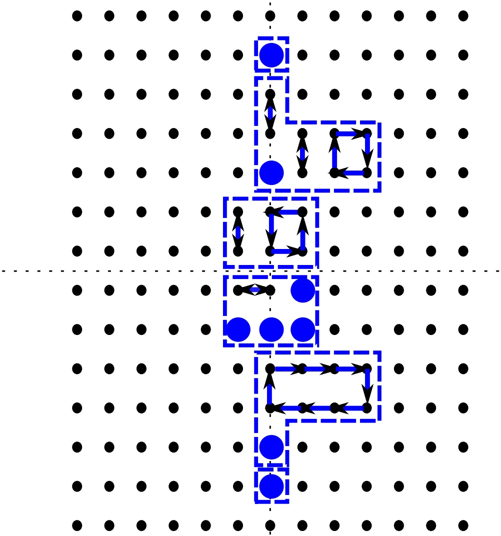

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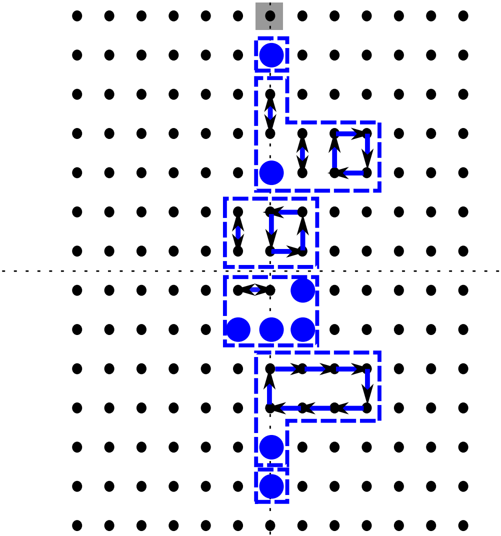

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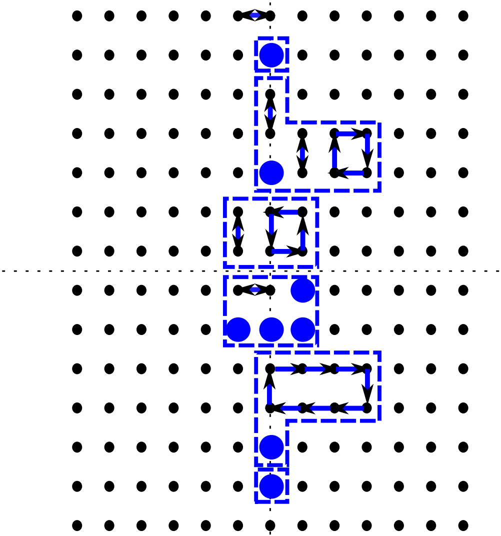

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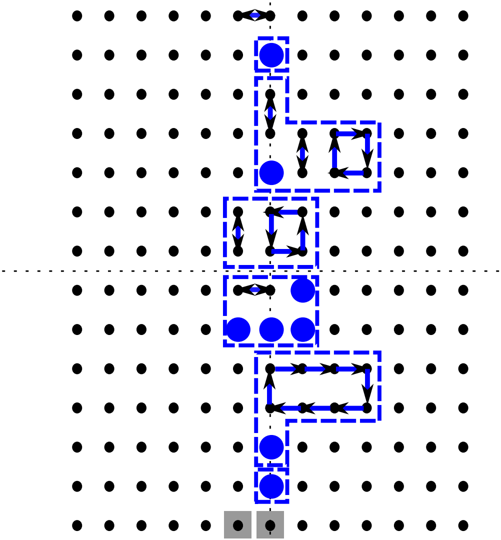

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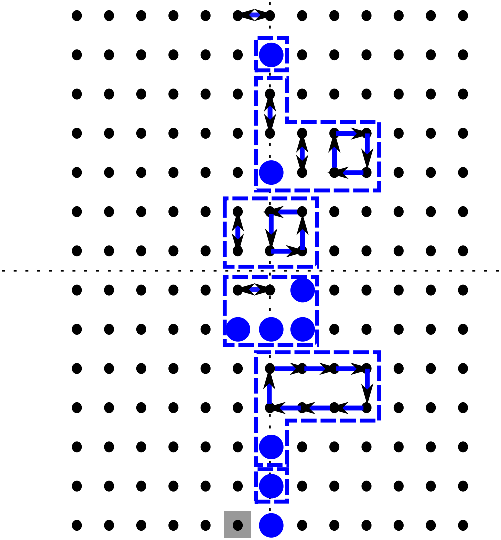

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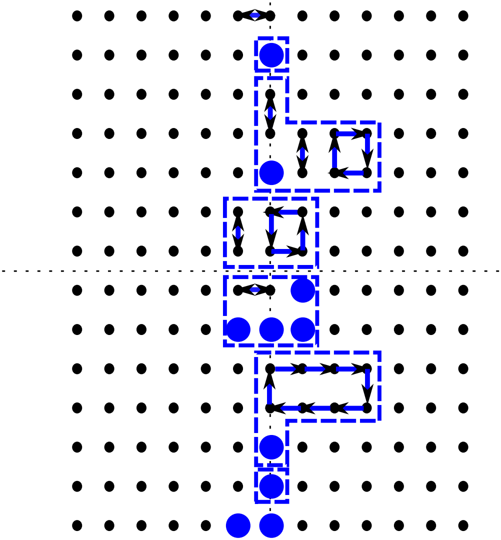

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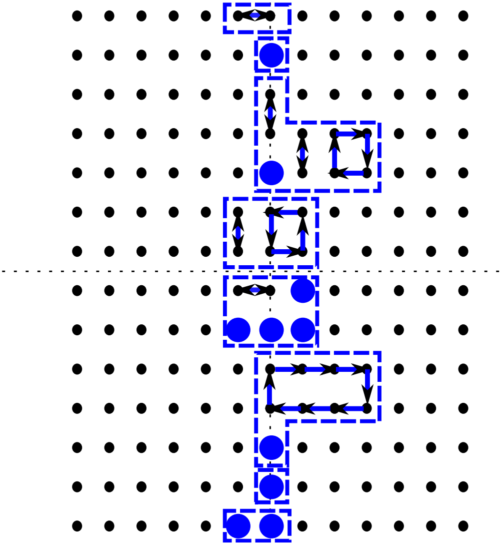

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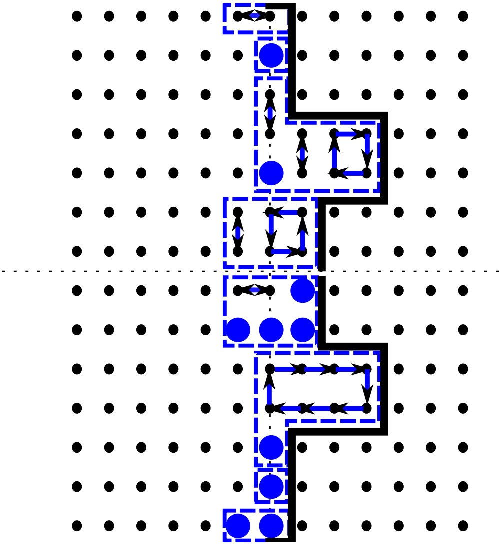

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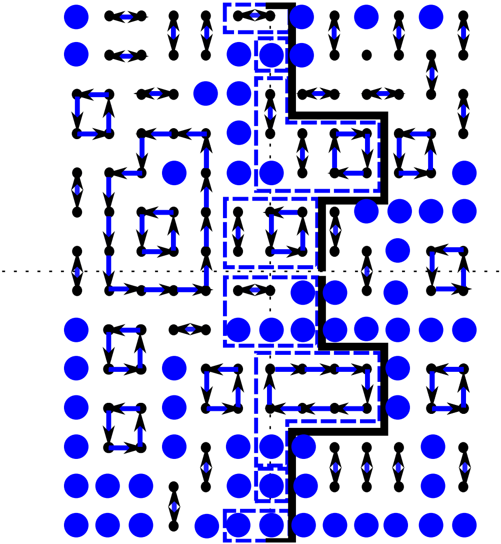
Last step: We sample $\sigma_{\Lambda \setminus \text{clusters}}$.



Lemma

If we put together the cycles that we kept at any step, we have a permutation which is distributed like $\mathbb{P}_\Lambda$

The procedure defines a stochastic process $\mathcal{N}_1$, $\mathcal{N}_2$, $\mathcal{N}_3$, ...

Cycle growth process

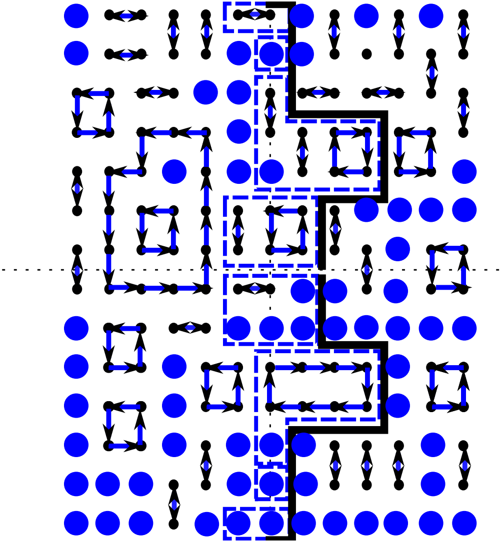
Proposition There exists $\alpha_c < \infty$ such that $\forall \alpha > \alpha_c$ the following holds. Namely, conditional on any realization of the procedure up to the step i, we have that $\forall n \in \mathbb{N}$,

$$P(|\gamma_{i+1}| > n \mid x_0, \sigma_\Lambda, \ldots x_i, \sigma_{B_i}) \leqslant C \exp\{-cn\},$$

where $c(\alpha), C(\alpha) > 0$, $|\gamma_{i+1}|$ cardinality of γ_{i+1}.

Theorem Let $(W_x)_{x \in \text{ vert line}}$ be a sequence of i.i.d. rand. var. distributed like the total population of a Galton-Watson process. Then, $\forall\, n \in \mathbb{N}$,

$$\mathbb{P}_\Lambda(\text{ max distance from vert. line} > n\ ) \leqslant P(\ \exists x \in \text{ vert. line } : W^x > n\ )$$



Open problems

1. convergence to Brownian motion under diffusive scaling?

2. understanding regime of small α

3. monotonicity with respect to α