High temperature regime in spatial random permutations

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Definition: Nearest neighbor SRP with forced long cycle

Set $\Lambda_L = [0, L]^d \cap \mathbb{Z}^d$ and fix two sites $z^1, z^2 \in \Lambda_L$

- **Forced open cycle between $z^1$ and $z^2$**
  \[ S_{\Lambda_L}^{z^1 \rightarrow z^2} = \text{set of bijections} \]
  \[ \pi : \Lambda_L \setminus \{z^1\} \rightarrow \Lambda_L \setminus \{z^2\} \]
  with the constraint that $\pi(x) = x$ or $\pi(x) \sim x$ for all $x \in \Lambda_L$.

- **Gibbs measure** depending on $\alpha \in [0, \infty)$
  \[ \mathbb{P}_\Lambda(\{\pi\}) = \frac{1}{Z(\Lambda)} \exp\left(-\alpha \sum_{x \in \Lambda} |\pi(x) - x|\right), \]

- **Actual settings**
  
  1. **forced open cycle between opposite sides.**
     
     $z_1 = \text{centre of the boundary side of the box}$,
     
     \[ S_{\Lambda_L} = \sum_{z^2 \in \text{opp. side to } z^1} S_{\Lambda_L}^{z^1 \rightarrow z^2} \]

  2. **only cycles**, namely nearest-neighbours permutations $\pi : \Lambda \rightarrow \Lambda$. 

Difficulties and results

- **Difficulties and challenges**
  1. no classical spin system (bijections introduce constraints)
  2. energy of the open cycle much smaller than the energy of the system
  3. no comparison between the law of \( \pi \) and the law of \( \pi \) conditional on some local features.

- **Results** There exists \( \alpha_c < \infty \) such that for all \( \alpha > \alpha_c \),
  1. No long cycles exist
  2. Exponential decay of correlations
  3. Orstein-Zernike behaviour for the forced open cycle

in any dimension of \( \mathbb{Z}^d \).
Orstein-Zernike method
Define a proper function on the sample space \( f(\pi) = \{(r_1, \xi_1), (r_2, \xi_2), \ldots (r_N, \xi_N)\} \), such that

- Markov process \((r_i, \xi_i) \rightarrow (r_{i+1}, \xi_{i+1})\),

\[
P(r_{i+1}, \xi_{i+1} \mid (r_1, \xi_1), \ldots (r_i, \xi_i)) = Q_L(r_{i+1}, \xi_{i+1}, r_i, \xi_i)
\]

- Regeneration surfaces are defined to be symmetric under a reflection with respect to \(r_i, 2\). This implies that

\[
E[(r_{i+1} - r_i) \cdot e_2 \mid r_i, \xi_i] = 0.
\]

for any \(r_i, \xi_i\).

- We prove that,

\[
P(|r_{i+1} - r_i| > D \log(L) \mid r_i, \xi_i) \leq C \exp\{-\sqrt{D} \; c\}
\]

with \(C\) and \(c\) independent on \(\xi_i\) and positive.
Orstein-Zernike method

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- Regeneration surfaces are defined to be symmetric under a reflection with respect to \(r_{i,2}\). This implies that

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with \(C\) and \(c\) independent on \(\xi_i\) and positive.
We define a set of indep. r.v. $(\sigma_A)_{A \subset \mathbb{Z}^d}$, where $\sigma_A$ is distributed like $\mathbb{P}_A$.

- **Step 1:**
  1. choose a site $x_1$ on the vertical line
  2. sample $\sigma_\Lambda$ and keep only the cycle intersecting $x_1$, $\gamma_1$
  3. define $B_1 = \Lambda \setminus \gamma_1$
  4. define the “non matching set” $N_1$

- **Step 2:**
  1. choose a site $x_2$ of the non-matching set $N_1$
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Cycle growth process

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We have the first cluster. We start again from a new site on the vertical line.
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**Last step:** We sample $\sigma_\Lambda \setminus \text{clusters}$.

**Lemma**
If we put together the cycles that we kept at any step, we have a permutation which is distributed like $\mathbb{P}_\Lambda$

The procedure defines a stochastic process $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \ldots$
Proposition There exists $\alpha_c < \infty$ such that $\forall \alpha > \alpha_c$ the following holds. Namely, conditional on any realization of the procedure up to the step $i$, we have that $\forall n \in \mathbb{N}$,

$$P(\vert \gamma_{i+1} \vert > n \mid x_0, \sigma_\Lambda, \ldots x_i, \sigma_{B_i}) \leq C \exp\{-cn\},$$

where $c(\alpha), C(\alpha) > 0, \vert \gamma_{i+1} \vert$ cardinality of $\gamma_{i+1}$.

Theorem Let $(W_x)_{x \in \text{vert line}}$ be a sequence of i.i.d. random variables distributed like the total population of a Galton-Watson process. Then, $\forall n \in \mathbb{N}$,

$$\mathbb{P}_\Lambda(\text{max distance from vert. line} > n) \leq P(\exists x \in \text{vert. line} : W^x > n)$$
Open problems

1. convergence to Brownian motion under diffusive scaling?

2. understanding regime of small $\alpha$

3. monotonicity with respect to $\alpha$