AN EXTREMAL PROPERTY OF DELAUNAY TRIANGULATION AND ITS APPLICATIONS IN MATHEMATICAL PHYSICS

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1. Delaunay triangulation

Let $\{P_i\}_{i=1}^n$ be a finite set of points in the plane. A set $\{D_j\}_{j=1}^m$ of triangles is called *triangle mesh* or *triangle tessellation* with knots $\{P_i\}_{i=1}^n$, if the following conditions are fulfilled:

a. The interiors of triangles are pairwise disjoint:

$$int\ D_j \cap int\ D_k = \emptyset, \quad j \neq k.$$

b. The set of all vertices of triangles is the set $\{P_i\}_{i=1}^n$.

c. The union of triangles fills the whole of convex hull of the knots:

$$\bigcup_{j=1}^m D_j = conv\{P_i\}_{i=1}^n.$$

A triangle mesh $\{D_j\}_{j=1}^m$ is called *Delaunay triangulation* with knots $\{P_i\}_{i=1}^n$, if the following condition is fulfilled:

d. For any triangle D_j

$$int\ S(D_j) \cap \{P_i\}_{i=1}^n = \emptyset, \quad j - 1, ..., m,$$

where $S(D)$ is the circumscribing circle of triangle D.

Let $T = \{D_j\}_{j=1}^m$ be a triangle mesh with fixed set of knots $\{P_i\}_{i=1}^n$. For a triangle D we denote by $A(D)$ the set of interior angles of D. Consider the following expression depending on mesh:

$$S(T) = \sum_{j=1}^{m} \sum_{\alpha \in A(D_j)} \cot \alpha. \tag{1.1}$$

Theorem 1. *For any point set the sum of cotangents* (1.1) *as a function on meshes with fixed set of knots reaches his minimum for Delaunay triangulation with same set of knots.*

Consider a convex tetragon with vertices A, B, C, D, see Fig. 1.

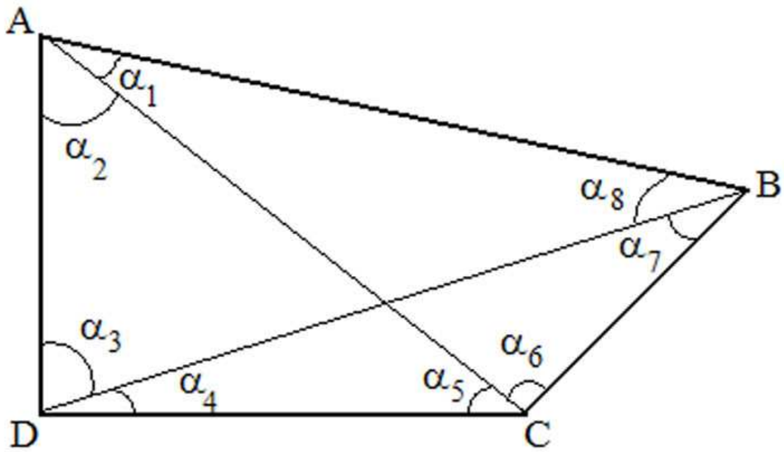

Fig. 1.

It is possible two triangle meshes with knots A, B, C, D. The first mesh T_{BD} with diagonal BD contains the triangles $\triangle ABD$ and $\triangle BCD$. The second mesh T_{AC} with diagonal AC contains the triangles $\triangle ABC$ and $\triangle ACD$. For mesh T_{BD} the sum of cotangents is

$$S(T_{BD}) = \cot(\alpha_1 + \alpha_2) + \cot \alpha_3 + \cot \alpha_8 + \cot(\alpha_5 + \alpha_6) + \cot \alpha_4 + \cot \alpha_7.$$

For mesh T_{AC} we have

$$S(T_{AC}) = \cot(\alpha_3 + \alpha_4) + \cot \alpha_2 + \cot \alpha_5 + \cot(\alpha_7 + \alpha_8) + \cot \alpha_1 + \cot \alpha_6.$$

If

$$\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 > \pi,$$

then the point D lies outside of the circumscribing circle of triangle $\triangle ABC$ and $S(T_{AC}) < S(T_{BD})$. Hence the mesh T_{AC} is Delaunay triangulation. If

$$\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 < \pi,$$

then the point C lies outside of the circumscribing circle of triangle $\triangle ABD$ and $S(T_{AC}) > S(T_{BD})$. Hence the mesh T_{BD} is Delaunay triangulation. In both cases the assertion of Theorem 1 is valid.

2. An application to numerical solution of equations system

Let $T = \{D_j\}_{j=1}^m$ be a triangle mesh with set of knots $\{P_i\}_{i=1}^n$. A knot P_i of mesh T can be a boundary knot, which belongs to the convex hull of T, or an interior knot. Let $I(T)$ be the set of interior knots of the mesh T. Denote by $V(D)$ the set of vertices of triangle D, and by E_i the set of triangles D from T connecting the knot P_i:

$$E_i = \{D \in T : P_i \in V(D)\}, \quad i = 1, ..., n.$$

Consider the following system of linear equations with respect to unknowns x_i:

$$\sum_{D \in E_i} \sum_{v \in V(D) \cap I(T)} \cot \alpha(v, D) x_k = b_i, \quad P_i \in I(T), \tag{2.1}$$

where $\alpha(v, D)$ is the interior angle of triangle D at the vertex v, k is the number of knot v, i.e. $v = P_k$.

Observe that the system (2.1) depends on the mesh T. We would like to solve the system (2.1) numerically by successive iterations method. The convergence rate of successive iterations depends on geometrical configuration of the mesh T. The well-known principle of diagonal domination says that the decrease of non-diagonal elements of corresponding matrix implies the increase of the convergence rate of successive iterations. Using Theorem 1 we can obtain the following assertion.

Lemma 1. *Among triangle meshes with the fixed set of knots, the maximal convergence rate of successive iterations to the numerical solution of the system* (2.1) *has the Delaunay triangulation with same set of knots.*

3. Applications to differential equations

The sum of cotangents arises by approximation of Laplace operator. For a triangle with vertices 1,2,3 we have the approximation formula

$$\nabla f \approx f(1)\cot\alpha_1 + f(2)\cot\alpha_2 + f(3)\cot\alpha_3,$$

where $f(i)$ is the value of function f at vertex i, α_i is the interior angle at vertex i.

Consider a system of two two-dimensional differential equations

$$L(A(x,y),\mu(x,y)) = 0, \tag{3.1}$$

$$\mu = F\left(A, \frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}, \frac{\partial^2 A}{\partial x^2} \dots\right), \tag{3.2}$$

where L is a differential operator acting on two depending variables A and μ. The dependence F between the main variable (potential) A and the secondary variable (medium function) μ is known. But the function F is complicate such that the substitution (3.2) in (3.1) reduces to a complicate equation, which is practically unsolvable.

Example. The Maxwell equations for magnetic field:

$$\frac{\partial}{\partial x}\frac{1}{\mu}\frac{\partial A}{\partial x} + \frac{\partial}{\partial y}\frac{1}{\mu}\frac{\partial A}{\partial y} = \delta, \tag{3.3}$$

$$\mu = P_6\left(\sqrt{\left(\frac{\partial A}{\partial x}\right)^2 + \left(\frac{\partial A}{\partial y}\right)^2}\right), \tag{3.4}$$

where A is desired potential function, μ is magnetic permeability of medium, $P_6(.)$ is a polynomial of 6 degree, δ is a given function.

We consider a class of two-dimensional differential equations (3.1), (3.2) satisfying the following conditions:

(a) If μ is piecewise constant, then the equation (3.1) is solvable.

(b) If A is piecewise linear, then μ is piecewise constant.

These conditions hold for the case, where the differential equations are reducible to a variation problem, i.e. there exists a functional $J(A, \mu(A))$, which possesses an extremum on the solution of differential equations (3.1), (3.2). Then (3.1) can be obtained from J by Euler variation formula.

The usual method for numerical solution of such problems is the finite elements method. The considered domain is divided into small triangles (elements). The function A is assumed linear within each triangle, and by condition (b) μ will be constant within each triangle. Then the search of extremum of functional J is reduced to the solution of a system of linear equations in unknown values of potential A at the vertices of triangles.

The usual scheme of solution is as follows. Starting from an initial piecewise constant μ_0, we obtain by the condition (a) the piecewise linear function A_1. Then from A_1 we obtain by the condition (b) the piecewise constant function μ_1. Then from μ_1 we get A_2 etc. If this process of successive approximations converges, then the limit function $A = \lim_{n\to\infty} A_n$ is desired (numerical) solution of the problem. Evidently, the equations (3.3), (3.4) satisfy the conditions (a) and (b).

Proposition 1. *The convergence rate of process of numerical solution of the problem* (3.1), (3.2) *by the finite elements method, depends on geometrical configuration of the mesh, in the cells of which the medium factor* μ *is constant.*

The application of finite elements method to the equations (3.3), (3.4) reduces to the solution of following system of linear equations (see [3], [5]):

$$\sum_{D \in E_i} \sum_{v \in V(D) \cap I(T)} \cot \alpha(v, D) \mu(D) x_k = b_i, \quad P_i \in I(T), \tag{3.5}$$

Using Lemma 1 we can obtain the following assertion.

Theorem 2. *For the problem* (3.3), (3.4) *for any fixed knots set the best mesh is Delaunay triangulation.*

Example. We solve a problem (3.3), (3.4) using Delaunay triangulation. The Figure 2 shows the solution, while the Figure 3 shows the corresponding mesh.

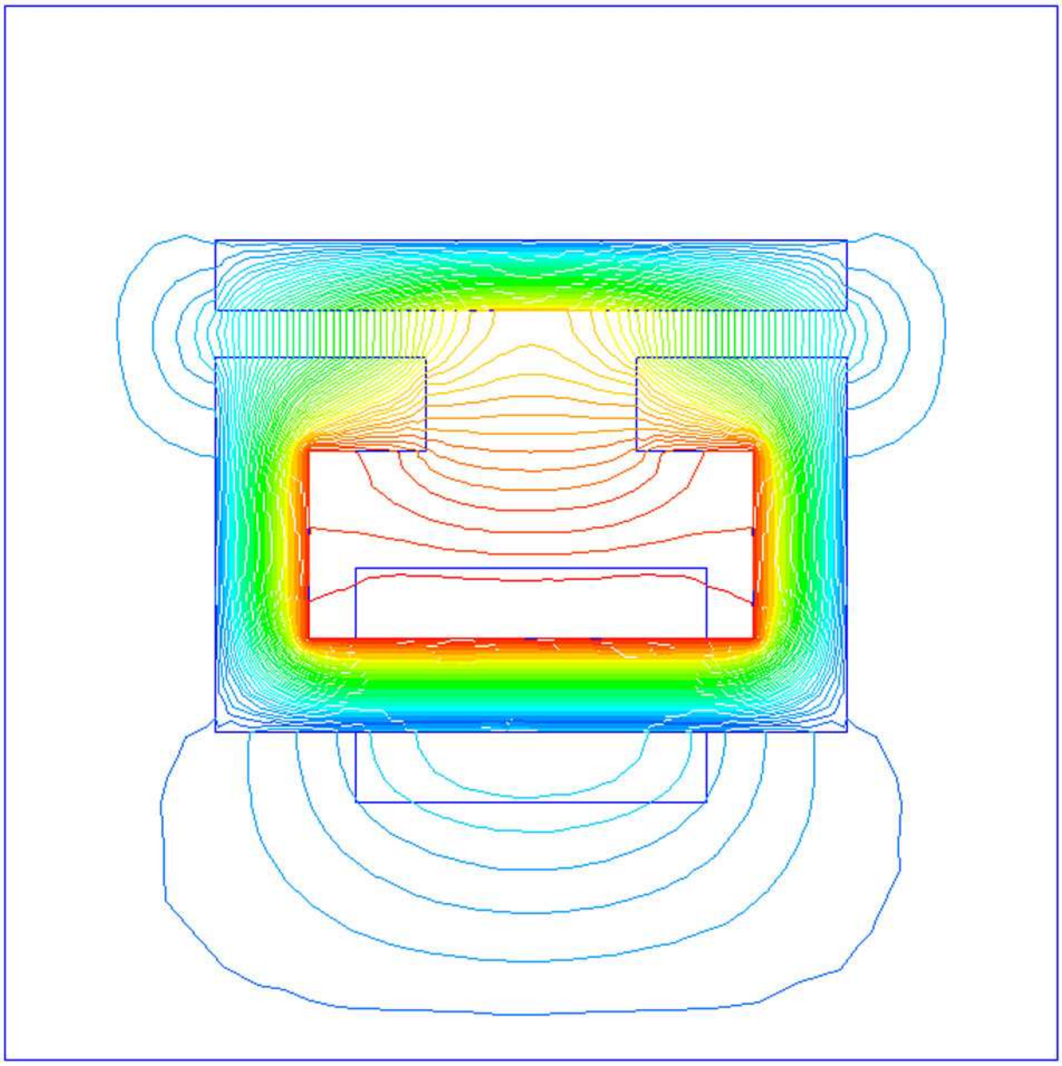

Fig. 2.

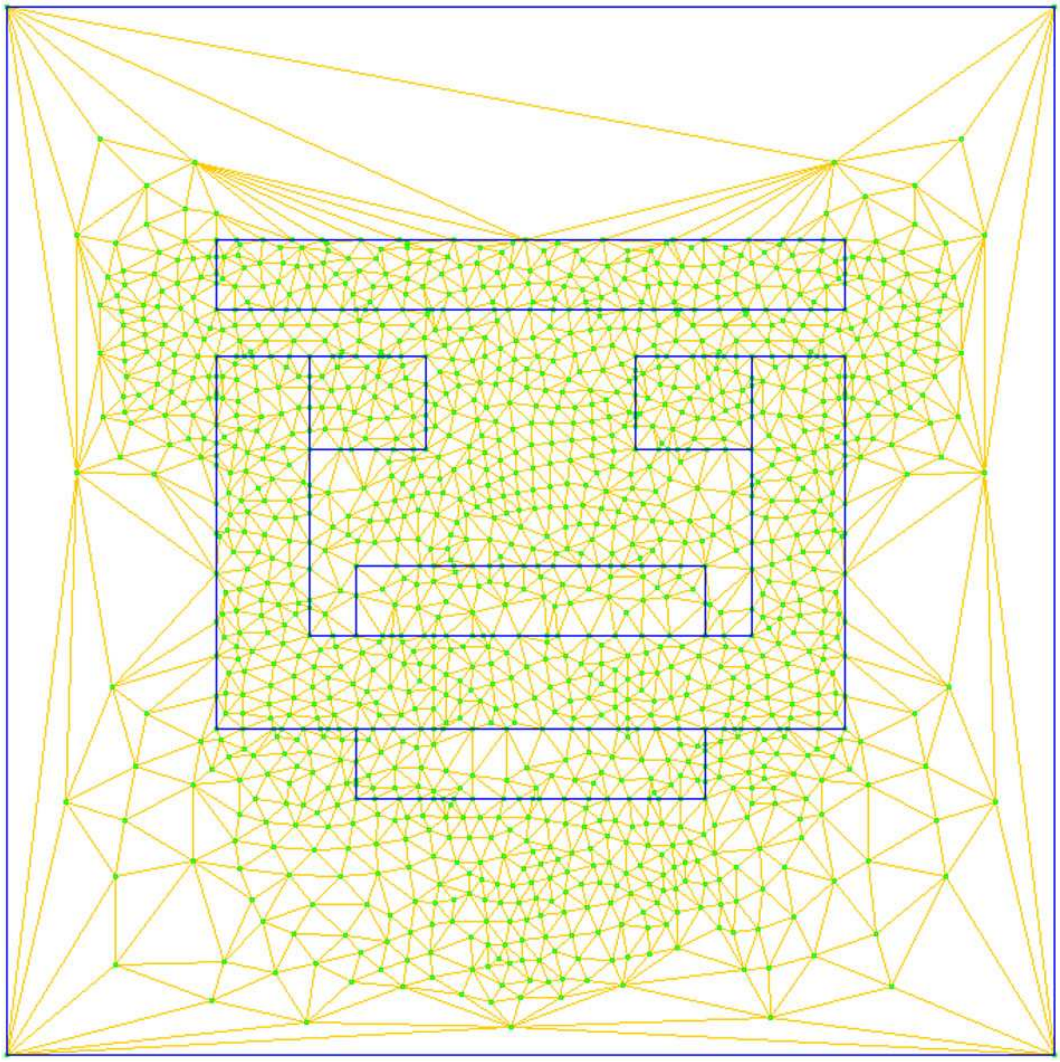

Fig. 3.

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